ON SURFACES ENVELOPED BY SPHERES BELONGING TO A LINEAR SPHERICAL COMPLEX*

BY

PERCEY F. SMITH

Introduction.

Surfaces enveloped by spheres cutting orthogonally a fixed sphere were first studied by MOUTARD† in the paper: *Sur la transformation par rayons vecteurs réciproques*, Nouvelles Annales de Mathématiques, ser. 2, vol. 3, 1864. In the discussion there given, the transformation of space named in the title is all important. The volume of DARBOUX, *Sur une class remarquable de courbes et de surfaces*, 1873, treats the surfaces and curves of MOUTARD more at length, with especial reference to the case of order 4, the well-known cyclides and bicircular quartics. These surfaces belong to the general class to be studied in this paper, viz.: Surfaces enveloped by spheres belonging to a linear spherical complex.

The configuration of ∞^3 spheres, the Kugelcomplex, owes its origin to SOPHUS LIE.‡ In the surfaces of MOUTARD the complex involved consists of all spheres intersecting a fixed sphere orthogonally. Other special cases are:

1°. Complex of all spheres of constant radius. The surface is now either a parallel or tubular surface.

2°. Complex of all spheres cutting a fixed plane under constant angle. This case has been treated in a paper by the author, *On a transformation of Laguerre*, Annals of Mathematics, ser. 2, vol. 1, July, 1900.

As in the investigations of MOUTARD and DARBOUX the familiar transformation, inversion in a sphere, serves as the main instrument, so in the following pages the properties of a more general contact-transformation under which spheres remain spheres, suffice for the derivation of the principal theorems.

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† These surfaces MOUTARD termed *surfaces anallagmatiques.*

‡ LIE, *Ueber Complexe* etc., Mathematische Annalen, vol. 5, 1872. The linear spherical complex is the aggregate of all ∞^3 spheres cutting a fixed fundamental sphere under a constant angle.

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This transformation depends upon five essential parameters and is an immediate generalization of spherical inversion. The transformation of Laguerre* and spherical inversion appear as equally justified particular cases of this general inversion. The relation of the most general contact transformation under which spheres remain spheres to this general inversion is investigated in §3.

It is believed that the results obtained in the paper are new, or, at all events, that the point of view has some novelty. In fact, the related investigations to which references are given below employ the line-sphere correspondence of Sophus Lie (loc. cit.), by which theorems in line geometry are translated into theorems on spheres. A direct method has, however, many advantages, for some of the relations of most importance in the geometry of spheres lose their peculiar metric significance when rendered into the language of line geometry; as, e. g., the second theorem of §3. Furthermore, the duality mentioned hold only for space of three dimensions, and therefore generalization is impossible. The present investigation moves entirely in the realm of the higher spherical geometry. It is to be regarded as analytical, and all equations may immediately be extended to the general case of n dimensions.

Treatment of the general case which shall be of a purely geometrical character is reserved for a later occasion. Means to this end are at hand in the (1, 1) correspondence of points in R_{n+1} and oriented spheres in R_n, the geometrical substratum of which is the so-called minimal projection.† There result in this way theorems of great simplicity and beauty.

In general contact transformations stress must be put upon the notion of *orientation*. A plane E to which a definite orientation, determined, e. g., by the direction of a normal to that plane, is assigned, may be called *oriented*. Analytically, such a plane, whose equation in rectangular current coördinates x, y, z is $Ax + By + Cz + D = 0$, is determined by the five homogeneous coördinates A, B, C, D, E,‡ satisfying identically

$$A^2 + B^2 + C^2 = E^2.$$

Planes tangent to the imaginary circle at infinity, which we shall call *minimum planes*, are characterized by $E = 0$, i. e., no distinction for orientation exists for such a plane. In fact, the normal to a minimum plane lies in that plane.

The aggregate of a point P and an oriented plane E through P constitutes an *oriented surface element*, and may be denoted by (P, E). If P is on a surface F, and E tangent to F at P, then (P, E) is said to belong to F.

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*SMITH, loc. cit.

†Cf. LIE, *Geometrie der Berührungstransformationen*, vol. I, p. 453, or KLEIN, *Höhere Geometrie*, p. 472.

‡With the oriented plane for element of space one has the geometry of reciprocal directions: SMITH, loc. cit., p. 170.

Thus a given surface defines two distinct continuous systems each of ∞^2 oriented surface elements. If, now, F is regarded as defined by one only of these two systems, F is said to be *oriented:* otherwise, *non-oriented.* A point P or a curve C likewise defines two systems of oriented surface elements: in the first instance, namely, the ∞^2 oriented surface elements with the same point P, and in the second, the aggregate of the (P, E)'s where P is on C and E a tangent plane to C at P. And now we have the definition of tangency fundamental for contact transformations:

Two oriented surfaces, two points or two curves, or a point and oriented surface, point and curve, or curve and oriented surface are tangent if they possess a common oriented surface element.

§ 1. *The linear spherical complex.*

In this section certain known results necessary to the present investigation will be recapitulated.*

In the higher spherical geometry of three dimensional space the element chosen is the *oriented* sphere. The point and oriented plane are degenerate cases. Analytically, such an element is determined uniquely by the ratios of six coördinates, called *hexaspherical coördinates*, $x_1, x_2, \cdots, x_6$, which satisfy identically a quadratic relation with non-vanishing discriminant. If the equation of a sphere is written in the form:

$$x_5(x^2 + y^2 + z^2) - 2x_1xt - 2x_2yt - 2x_3zt - 2x_6t^2 = 0,$$

x, y, z, t being current rectangular coördinates, and the radius r is made equal to x_4/x_5, the coördinates x_i satisfy the simple relation:

$$\phi(x) \equiv x_1^2 + x_2^2 + x_3^2 - x_4^2 + 2x_5x_6 = 0.$$

"Tangency" of two elements x_i and y_i is defined by the vanishing of the polar form:

$$\phi(x, y) \equiv x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 + x_5y_6 + x_6y_5.$$

If we adopt the convention: the normals to the surface of an oriented sphere are directed outward or inward according as the radius is greater or less than zero, then the condition of tangency becomes that given in the introduction, viz., the possession by the two elements in question, x_i and y_i, of a common oriented surface element. In addition, we deduce:

Two distinct points are tangent if they lie upon a minimum line.

Two oriented planes are tangent if their intersection is a minimum line.

* The dissertation of VIRGIL SNYDER, *Ueber die linearen Complexe der Lie'schen Kugelgeometrie*, Göttingen, 1895, gives a detailed purely geometrical discussion from the standpoint of the line-sphere transformation.

Analytically, the linear spherical complex is defined as the aggregate of the ∞^3 oriented spheres whose coördinates x satisfy the linear equation

$$\sum_i a_i x_i = 0 \qquad (i=1, 2, \cdots, 6). \tag{1}$$

The geometrical definition appears as follows:

Let x_i and y_i be two oriented spheres s and s' of radii r and r', d the distance of their centers and let the number κ be defined by the equation

$$d^2 - r^2 - r'^2 + 2rr'\kappa = 0. \tag{2}$$

If $|\kappa| < 1$, s and s' intersect under the angle $\cos^{-1}\kappa$.

In the coördinates the equation just written becomes

$$x_1y_1 + x_2y_2 + x_3y_3 - \kappa x_4y_4 + x_5y_6 + x_6y_5 = 0. \tag{3}$$

Comparing equations (1) and (3) we see that the variable sphere x stands in the relation indicated by (2) to the fixed sphere

$$\left(a_1, a_2, a_3, -\frac{a_4}{\kappa}, a_6, a_5\right),$$

where

$$\kappa^2 = \frac{a_4^2}{\phi(a) + a_4^2}.$$

For $|\kappa| < 1$, we have the usual statement, viz., the variable sphere intersects the fixed sphere under a constant angle.

We call the fixed sphere the *fundamental sphere* and κ the *constant* of the complex. The radius r of the fundamental sphere is determined from

$$r\kappa = -\frac{a_4}{a_6}, \quad \text{or} \quad r^2 = \frac{\phi(a) + a_4^2}{a_6^2}.$$

To the complex belong ∞^2 points, viz., the points of the fundamental sphere, and ∞^2 oriented planes, viz., the planes enveloping the sphere of radius $r\kappa$ concentric with the fundamental sphere.

The following particular cases are important:

(1) *Orthogonal complex*, $a_4 = 0$, consisting of all spheres cutting the fundamental sphere orthogonally; here $\kappa = 0$. Such a complex will be denoted by I, I', etc.

(2) *Plane complex*, $a_6 = 0$, the fundamental sphere degenerating into an oriented plane. If r is the radius of any sphere of the complex and p the perpendicular distance of its center from the fundamental plane, then the constant $\kappa = p/r$. The fundamental plane has for its equation:

$$a_1x + a_2y + a_3z + a_5t = 0,$$ *

* For a detailed exposition of the properties of the plane complex, cf. SMITH, loc. cit.

and is therefore at infinity if $a_1 = a_2 = a_3 = 0$. The complex is now the aggregate of all oriented spheres of radius $-a_5/a_4$. A general plane complex will be symbolized by E, the special case just mentioned by D.

(3) *Special complex*, $\phi(a) = 0$. The ∞^3 spheres of the complex are tangent to the fundamental sphere $(a_1, a_2, a_3, -a_4, a_6, a_5)$, since $\kappa = 1$.

(4) *Complex of all points of space*, $a_i = 0$, $i + 4$. This complex is general, since $\phi(a) = -1$. The coördinates of the fundamental sphere vanish and the latter becomes illusory. The symbol Ω will be used for this complex.

(5) *Complex of all oriented planes of space*, $a_i = 0$, $i + 5$. This complex is *special*, the fundamental sphere being the plane at infinity.

Two complexes A and B are said to be *conjugate* or *in involution* when the polar form

$$\phi(a, b) \equiv a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4 + a_5b_6 + a_6b_5$$

vanishes. If r and r' are respectively the radii of the fundamental spheres s and s', d the distance of their centers, and κ and κ' the constants of the complexes, the condition for involution is

$$d^2 - r^2 - r'^2 + 2rr'\kappa\kappa' = 0.$$

If, however, A is a plane complex, this becomes

$$p + \kappa\kappa' = 0,$$

where p is the perpendicular distance from the center of s' to the fundamental plane. Finally, if A and B are both plane complexes and θ the angle between their fundamental planes, the condition of involution is

$$\cos\theta + \kappa\kappa' = 0.$$

The following results are easily derived:

Every complex conjugate to the complex of all points is orthogonal.

Every complex conjugate to the complex of all planes is a plane complex.

If one of two complexes in involution is orthogonal, the fundamental spheres intersect orthogonally.

If one of two complexes in involution is special, its fundamental sphere belongs to the other complex.

Consider now five orthogonal complexes I_k, $k = 1, 2, \cdots, 5$, mutually in involution. Since the complex of all points Ω is conjugate to each of these, the six complexes Ω, I_k, constitute a system of mutually conjugate complexes. Systems of six general complexes mutually conjugate may be constructed.

A degenerate system arises when one of the complexes is special. In this case the special complex is to be reckoned double and its fundamental sphere belongs to each of the remaining four. An interesting case occurs if this special

complex is the complex of all planes of space, for then the other four complexes are plane complexes.*

§ 2. *Inversion in a spherical complex: definition and properties.*

By a linear transformation

$$\rho A_i' = \sum_k a_{ik} A_k \qquad (i, k = 1, 2, \cdots, 6),$$

the spherical complex A transforms into the complex A'. We shall consider only such transformations as carry a special complex over into a special complex, i. e., the a_{ik} are assumed such that

$$\phi(A') \equiv \lambda \phi(A).$$

Involution is now an invariantive property of the transformation, and the latter depends upon fifteen essential parameters. When A is special, the A_i are related to the coördinates x_i of the fundamental sphere by the ratios

$$A_1 : A_2 : A_3 : -A_4 : A_6 : A_5 :: x_1 : x_2 : x_3 : x_4 : x_5 : x_6. \tag{4}$$

With these substitutions and correspondingly for A_i', the transformation in question becomes *a contact transformation under which spheres remain spheres.*

Consider in particular the transformation of five essential parameters

$$a_1 : a_2 : a_3 : a_4 : a_5 : a_6, \qquad (\phi(a) \neq 0),$$

defined by

$$A_i' = A_i - \frac{2\phi(A, a)}{\phi(a)} a_i. \dagger \tag{5}$$

Then $\phi(A', a) + \phi(A, a) = 0$, and by the substitutions (4), $\phi(A, a)$ becomes $\sum_k a_k x_k$, $(k = 1, 2, \cdots, 6)$. The transformation (5) has, therefore, the following properties:

A sphere belonging to the complex $\sum_\kappa a_\kappa x_\kappa = 0$ *is invariant; every sphere of this complex which is tangent to an arbitrary sphere* x *is tangent also to the transformed sphere* x', *and reciprocally.*

These theorems furnish a purely geometrical definition of the transformation.

The transformation is uniquely determined by the complex a, and conversely. We shall call the transformation *inversion in the linear spherical complex* a.‡

The fundamental sphere of a is the *fundamental sphere* of the transformation, and its center the *center* of the transformation. We shall indicate the inversion in the complex a by (a).

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*SMITH, loc. cit., p. 165.

† The determination of (5) is simply this: A_i' shall have the form $\lambda A_i + \mu a_i$.

‡ Cf. the memoir of E. MÜLLER, *Ueber die Geometrie orientirter Kugeln* etc., Monatshefte für Mathematik und Physik, 1898, p. 314. Under the line-sphere transformation this corresponds to the dualistic inversion in a line complex.

Particular cases are the following:

When a is an orthogonal complex, (a) is the well known spherical inversion in the fundamental sphere.

When a is a plane complex, (a) is the transformation by reciprocal directions of Laguerre.*

When a is a D, say $(0, 0, 0, 1, \frac{1}{2}c, 0)$, corresponding spheres under (a) are concentric and their radii r and r' satisfy the relation $r' + r = -c$, i. e., (D) is a *dilatation*.

When a is the complex of all points Ω, (Ω) merely changes orientation.

The fact of the transformation of the complex A under (a) into A' will be indicated by

$$A(a)A',$$

or, since (a) is involutory,

$$A'(a)A.$$

We now establish certain theorems.

THEOREM 1. *A complex A not a plane complex may be transformed by a unique dilatation (D) into an orthogonal complex I.*

In fact, by inversion in

$$D: (0, 0, 0, A_6, \tfrac{1}{2}A_4, 0),$$

A becomes an I:

$$\left(A_1, A_2, A_3, 0, \frac{\phi(A) - A_1^2 - A_2^2 - A_3^2}{2A_6}, A_6\right).$$

The spherical inversion (I) has the same center as (A), and its power, i. e., the square of the radius of its fundamental sphere, is $R^2(1-\kappa^2)$, where R is the radius of the fundamental sphere and κ the constant of A. The radius of the spheres of D is $-\frac{1}{2}R\kappa$. If A is orthogonal, then D is Ω, and I and A are identical.

THEOREM 2.—*If $A(D)I$, then $\Omega(D)D'$, where D' is $(0, 0, 0, A_6, A_4, 0)$, and D' and A are in involution.*

This easily follows, since Ω and I are in involution.

THEOREM 3.—*Any complex A may be transformed by a spherical inversion (I) into a plane complex, and any pencil into a pencil of plane complexes.*

For if $a_4 = 0$, A becomes by (a) a plane complex if the a's satisfy

$$A_6\phi(a) - 2\phi(A, a)a_6 = 0.$$

In fact, when the center of (a) is chosen in the common intersection of the fundamental spheres of A and B, then every complex $\lambda A + \mu B$ inverts into a plane complex.

* Cf. SMITH, loc. cit., p. 157.

If $A(a)A'$, then the inversion (A) is transformed by (a) into (A'). Thus the theorems take the form:

1) $$(A) = (DID);$$

2) $$(D') = (D\Omega D);$$

3) $$(A) = (I'EI').$$

Hence: The general inversion is uniquely compounded of a dilatation, a spherical inversion, and the same dilatation, or of a spherical inversion, inversion in a plane complex, and the same spherical inversion, the latter resolution not being unique.

Since, now, by an (I), two general complexes A and A' may be transformed into plane complexes, E and E', then

$$(AA') = (IEE'I).$$

Theorems established for the transformation (EE')* are thus transformed into theorems on the transformation (AA'), and we may state the following results:

THEOREM 4.—*The transformation of space which is compounded of successive inversions in two spherical complexes may be resolved in ∞^1 ways into successive inversions in two other spherical complexes. The four complexes belong to the same pencil; and, of the two latter, one may be chosen arbitrarily, and the other is then uniquely determined.*

Since in the pencil $\lambda A + \mu A'$ there is at least one orthogonal complex I and one plane complex E, the transformation (AA') is identical with a spherical inversion (I) or inversion in a plane complex (E) followed by a second inversion, or an inversion compounded with the same (I) or (E); i. e.,

$$\begin{aligned}(AA') &= (IB) \;\; = (B'I)\\ &= (EC) = (C'E).\end{aligned}$$

THEOREM 5.—*The inversions (A) and (A') are commutative when, and only when, the complexes A and A' are in involution. Then (AA') is an involutory transformation.*†

An important case is where (A) and (A') are commutative and have the same fundamental sphere σ. One has then

$$\kappa\kappa' = 1.$$

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* Cf. SMITH, loc. cit., p. 169, seq., and the paper by WIENER: *Ueber die aus zwei Spiegelungen zusammengesetzten Verwandtschaften*, Leipziger Berichte, 1891, p. 660.

† It is easy to show that $(AA') = (BB')$ when and only when

$$\frac{\varphi(A, A')}{\sqrt{\varphi(A)\varphi(A')}} = \frac{\phi(B, B')}{\sqrt{\phi(B)\phi(B')}}.$$

If (I) is the spherical inversion in σ, then

$$(AA') = (I\Omega) = (\Omega I),$$

i. e.,

$$(A) = (A'\Omega I) = (I\Omega A').$$

Inversion in a complex with constant κ may therefore be resolved into inversion in a complex with the same fundamental sphere and constant equal to $1/\kappa$, change in orientation and inversion in that sphere.

THEOREM 6.—*There are two complexes C_1 and C_2 such that for any two inversions (A) and (B)*

$$(A\ C_1) = (C_1 B),$$

$$(A\ C_2) = (C_2 B),$$

and then

$$(C_1 C_2) = (C_2 C_1);$$

that is, any two inversions may be transformed into each other by two inversions, and these two inversions are commutative.

For example, the complex A is transformed into Ω by inversion in

$$A_0\colon (A_1,\ A_2,\ A_3,\ A_4 \pm \sqrt{-\phi(A)},\ A_5,\ A_6),$$

i. e., *a spherical complex A may be inverted into the complex of all points by two inversions for each of which the fundamental sphere is identical with the fundamental sphere of A.*

If κ_0 is the constant of A_0, then $\kappa_0^2 - 2\kappa\kappa_0 + 1 = 0$. The complexes A_0 and A_0' are therefore in involution. In particular, if A is a *plane* complex, so also is A_0. Again, by theorems 1 and 2

$$(AD') = (D'A).$$

If, therefore, S is a sphere of A and

$$S(A)S(D')S',$$

then since

$$S(D')S'(A)S',$$

S' also belongs to A, and *the spheres of A are concentric in pairs and inverse with respect to (D').* When A is orthogonal, S and S' differ only in orientation.

THEOREM 7.—*If $(AD') = (D'A)$ and $(A) = (A_0\Omega A_0)$, then by (A_0), (D') is transformed into a spherical inversion (I'), and the concentric spheres of A, viz., S and S', invert by (A_0) into two points inverse with respect to (I').*

Finally we may add this theorem:

THEOREM 8.—*By a general inversion* (A_0) *which transforms* (Ω) *into* (A) *the system of six mutually conjugate complexes consisting of* Ω *and five orthogonal complexes whose fundamental spheres intersect orthogonally is inverted into a general system of six conjugate complexes of which* A *is one. In the same way the degenerate system made up of the complex of all planes and four conjugate plane complexes transforms into a general degenerate system.*

§ 3. *The group* G_{15} *of all contact transformations which transform spheres into spheres.*

By successive applications of theorem 4 of the preceding section, we have:

A transformation (T) *compounded of any number* n *of inversions may be resolved into one inversion followed by* $n-1$ *spherical inversions, or* $n-1$ *spherical inversions followed by an inversion.*

If now n is odd, by theorem 5 we may replace the inversion in question so that the number of spherical inversions shall always be odd. But an odd number of spherical inversions reduces to a displacement and one spherical inversion, or to one spherical inversion followed by a displacement; hence the theorem:

The transformation (T) *may be reduced to one of the forms*

$$(a) \qquad (AIS) = (S'I'A'),$$

$$(b) \qquad (A\Omega IS) = (S'I'\Omega A'),$$

where S *is a displacement.* Or, since a displacement is compounded of at the most four symmetries, *the transformation* (T) *may be reduced to successive inversions in six and seven spherical complexes respectively.**

The transformation (T) depends on 15 essential parameters and is the most general contact transformation which transforms spheres into spheres. For, analytically, the most general transformation of this kind is the linear transformation

$$\rho x_i' = \sum_k a_{ik} x_k \qquad (i, k = 1, 2, \cdots, 6),$$

where

$$\phi(x') \equiv \mathrm{K}\phi(x),$$

and it is easy to resolve this into successive inversions. The ∞^{15} transformations form two continuous systems, of which (a) and (b) are typical forms. The transformations (a) of themselves form a continuous group, and the entire set of (a) and (b) give the group G_{15} of all contact transformations under which spheres remain spheres.

* A transformation of the type (a) may be made to assume the form $((A_1A_2)(A_3A_4)(A_5A_6))$, where A_1 and A_2, A_3 and A_4, A_5 and A_6 are in involution. That is, (T) in this case may be resolved into three involutory transformations. Then by the line sphere duality, any collineation may be resolved into three skew reflections. Cf. E. B. WILSON, Transactions of the American Mathematical Society, vol. 1, p. 193, 1900.

In the Mathematische Annalen, vol. 5, p. 186, SOPHUS LIE gives the theorem that every transformation of the type here considered is compounded of dilatations and spherical inversions. Since, by theorem 1, $(A) = (DID)$, the truth of LIE's result is established.

The resolutions (a) and (b) are, it is believed, new, and show that what is essentially novel in such a transformation can be learned by the study of the inversion in one general complex.

§ 4. *Surface sibi-reciprocal under a general inversion.*

In this section the problem under discussion is attacked.

The transformation of a surface F into a surface F' by (A) is effected as follows: Let (P, E) be an oriented surface element of F, and let P and E by (A) become the oriented spheres s and s'; then s and s' have a common oriented surface element (P', E'), and this belongs to F'. We note that F' is necessarily an oriented surface. If then F under (A) transforms into itself, F must in general be oriented. In particular: a non-oriented surface is sibi-reciprocal under (Ω), and conversely, if

$$F(\Omega)F,$$

then F is non-oriented.

Let now $F(A)F$, and consider any sphere s of A tangent to F; then since $s(A)s$, *s must touch F again;* i. e., *the surface F is the envelope of spheres belonging to A.*

It follows from theorem 6 of § 2, that if $(A_0) = (A\Omega A)$ and $F(A_0)f$, then $f(\Omega)f$, and f is *an ordinary point locus.* The result is

THEOREM 1.—*A surface enveloped by spheres belonging to a linear spherical complex may always be derived by a general inversion from a primitive curve or surface regarded as a point locus.*

This theorem furnishes the point of view for the following. If the primitive is a curve, the derived surface is an *annular* surface, one system of lines of curvature being circles.

We now establish various theorems by the aid of the results of § 2.

By theorem 1 of § 2, if $(A_0) = (D_0 I_0 D_0)$, then let, since $f(A_0)F$,

$$f(D_0)f'(I_0)f''(D_0)F,$$

i. e., the surface and its primitive are transformed by the same dilatation (D_0) into surfaces inverse under a spherical inversion concentric with (A).

From the known properties of dilatations* and spherical inversions, we deduce the following theorems, under the assumption that the center of inversion has *general* position with respect to the primitive:

*Cf., e. g., the paper by S. ROBERTS, Proceedings of the London Mathematical Society, 1873, p. 94.

If the primitive is a surface of order m, class n, and if $2N$ is the order of its non-singular minimum developable,* then the derived surface is of order and class $2(2m - n + 2N)$ and passes $2m - n + 2N$ times through the imaginary circle at infinity.

If the primitive is a curve of order m, rank r, passing $2p$ times through the imaginary circle at infinity, then the derived annular surface is of order and class $2(2m + r - 4p)$ and passes $2m + r - 4p$ times through the imaginary circle at infinity.

From these results we may state

THEOREM 2.—*In general, if a surface is enveloped by spheres belonging to a linear spherical complex, the order and class are equal, the surface intersects the plane at infinity along the imaginary circle, and the minimum developable is singular.*

The points of the fundamental sphere being unchanged by (A_0), we have

THEOREM 3.—*The curve of intersection of the primitive surface and the fundamental sphere is a double line on the derived surface. The points of intersection of the primitive curve and fundamental sphere are conical points of the derived annular surface.*†

We may now obtain another theorem from the resolution of (A_0) (theorem 3, §2):

$$(A_0) = (I'_0 E_0 I'_0).$$

Let $f(I'_0)f'$, and $F(I'_0)F'_0$; then since $f(A_0)F$, we must have

$$f'(E_0)F'_0.$$

From the properties of the inversion (E_0), if the point P on f' transform by (E_0) into the oriented sphere s touching F'_0 in P' and P'', then the circle passing through P, P' and P'' intersects f' and F'_0 orthogonally, and its center lies in the fundamental plane of E_0.‡ Transforming back by (I_0) we get

THEOREM 4.—*If the point M on f invert by (A_0) into the sphere S, then the circle drawn through M orthogonal to f and the fundamental sphere of (A_0) intersects S and F orthogonally in the points of contact of S and F.*§

The results just deduced are based upon the fact that a linear spherical complex A not special may be transformed by inversion in a complex A_0 into the

* The singular minimum developable of a surface containing the imaginary circle at infinity is the developable formed by planes tangent to the surface along this line. If f has no minimum developable except the singular developable, as is the case for the sphere, then $N = 0$.

† Interesting exceptions to the theorems 2 and 3 are noted below in §8.

‡ SMITH, loc. cit., p. 161, or DARBOUX, *Leçons sur la théorie générale des surfaces*, vol. I, p. 261.

§ Thus by the general theorem of RIBAUCOUR (cf. DARBOUX, loc. cit., vol. II, p. 330), the primitive surface f and the derived F belong to the same cyclic system, which may be obtained from f by inversions in the fundamental sphere with varying constant κ.

complex of all points Ω. We now write the equations of the transformation, using the defining formulas (5) of § 2. If the origin is taken for the center of the inversion, then $a_1 = a_2 = a_3 = 0$.

Denoting the radius of the fundamental sphere by r_0 and the constant of A_0 by κ_0, the point (x, y, z) inverts into the oriented sphere with center (x', y', z') and radius r' such that

$$(6) \qquad \begin{cases} x' = \dfrac{sx}{x^2+y^2+z^2-d^2}, & y' = \dfrac{sy}{x^2+y^2+z^2-d^2}, \\ z' = \dfrac{sz}{x^2+y^2+z^2-d^2}, & r' = \dfrac{d[r_0^2-(x^2+y^2+z^2)]}{x^2+y^2+z^2-d^2}, \end{cases}$$

where * $d = -r_0\kappa_0$, $s = r_0^2(1-\kappa_0^2)$.

These equations show that the ∞^2 points of the sphere $x^2+y^2+z^2=d^2$ correspond to the ∞^2 oriented planes of the complex A. We have already seen that such planes touch a sphere concentric with the fundamental sphere. Points of the primitive, therefore, lying on the sphere $x^2+y^2+z^2=d^2$ go over into bitangent or singular tangent planes to the derived surface.

These considerations give

THEOREM 5.—*If a surface is enveloped by ∞^2 spheres belonging to a linear spherical complex A, then a developable, whose planes belong to A and touch a sphere concentric with its fundamental sphere, has double contact with that surface.*

The line joining a point on the sphere $x^2+y^2+z^2=d^2$ and the center is normal to the corresponding plane. In fact, the curve of intersection of this sphere with the primitive surface and the developable of bitangent planes are reciprocal under a polar reciprocation in the center. The modification of these results for an annular surface is obvious.

Let now the primitive be a surface of order m passing r times through the imaginary circle at infinity. Then the derived surface contains a double curve (theorem 3) of order $2(m-r)$ lying on the fundamental sphere, while the developable having double contact with the surface (theorem 5) is the polar reciprocal of a curve of the same order. If the primitive is a curve of order m passing $2p$ times through the imaginary circle, then the derived annular surface possesses $2(m-p)$ conical points and the same number of singular tangent planes.

We pass now to the consideration of the *locus of centers* of the spheres enveloping the surface under discussion. The equations (6) give the point trans-

* Thus is established a (1, 1) involutory correspondence between the points of space and spheres of a linear spherical complex. The study of this correspondence is reserved for a later occasion. Geometrically, the point is a limiting point of the coaxial system determined by the corresponding sphere and the fundamental sphere of the complex.

formation of the primitive into this locus.* This is a $(1, 2)$ correspondence, a point (x', y', z') yielding two points (x, y, z) inverse under a spherical inversion (I') on the origin (theorem 7, § 2).

In general, then, the transformation of the primitive into the locus of centers is birational.† Corresponding points lie on a line through the origin and the imaginary circle at infinity is the "fundamental curve." We easily derive

THEOREM 6.—*If the primitive is a surface of order m, class n, passing r times through the imaginary circle at infinity, the derived surface is enveloped by ∞^2 spheres whose centers lie upon a surface of order $2(m-r)$, class $2m+n-4r$, having at the center of inversion a conical point of order $m-2r$.*

If the primitive is a curve of order m passing $2p$ times through the imaginary circle at infinity, the derived annular surface is enveloped by ∞^1 spheres whose centers lie upon a curve of order $2(m-p)$ passing $m-2p$ times through the center of inversion.

Let F denote the surface under discussion, i. e., $F(A)F$. Since by theorem 1, § 2, we have $(A) = (DID)$, if $F(D)F'$, then necessarily $F(I)F'$, or the surface F is transformed by a dilatation into a surface F' sibi-reciprocal under a spherical inversion on the center of A, and having the same locus of centers.‡ Let us call this locus of centers F_0, and the center of inversion O.

The points of contact of S, a sphere of A, with F are now easily constructed: viz., let E be a tangent plane to F_0 at P, the center of S, and let S' be the sphere of I with center P. Then S' touches F' in the points of intersection of S' and a line drawn through O perpendicular to E. The points of contact of S with F lie on the lines joining these points on S' with P. (Cf. theorem 4.) If, now, E is a minimum plane, then S' touches F' on the imaginary circle at infinity, and the same is true of S and F. Hence

THEOREM 7.—*The minimum developable of the locus of centers is the singular minimum developable of the surface F.*

§ 5. *The general inversion as a transformation of oriented surface elements.*

If the equation of a plane be written in current homogeneous rectangular coordinates x_1, x_2, x_3, x_4 § in the form

$$(ux) \equiv u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0,$$

* This transformation for the case $s = 2r_0^2$, $d^2 = -r_0^2$ is given by DARBOUX, *Sur une classe remarquable* etc., p. 123, and loc. cit., vol. III, p. 493, and appears below, § 8.

† Exception discussed below, § 8.

‡ Cf. DARBOUX, *Sur une classe* etc., p. 255.

§ The plane at infinity is supposed to be $x_4 = 0$.

and if u_0 be defined by $u_0^2 = u_1^2 + u_2^2 + u_3^2$, then the ratios

$$x_1 : x_2 : x_3 : x_4, \quad u_0 : u_1 : u_2 : u_3 : u_4$$

will determine a unique *oriented surface element* (x, u).

The construction given at the end of the previous section for the surface element corresponding to a given surface element (P, E) enables us to deduce the equations of inversion in the complex A_0 in terms of the above coördinates. Taking the origin for center of inversion, κ_0 for the constant of A_0, r_0 for the radius of the fundamental sphere, we easily derive for the transformation of (x, u) into (x', u') by (A_0) the following equations:

$$(7) \qquad \begin{cases} \rho x_i' = x_i\psi - (x_i u_0 - x_4 u_i d)(U - r_0^2 x_4^2) & (i=1, 2, 3), \\ \rho x_4' = x_4\psi, \\ \sigma u_0' = \psi, \\ \sigma u_i' = u_i(U - d^2 x_4^2) + 2x_i x_4(u_4 + d u_0) & (i=1, 2, 3), \\ \sigma u_4' = u_4(U - d^2 x_4^2) - (U + r_0^2 x_4^2)(u_4 + d u_0), \end{cases}$$

where

$$\psi \equiv u_0(U + d^2 x_4^2) + 2u_4 d x_4^2,$$

$$U \equiv x_1^2 + x_2^2 + x_3^2,$$

$$d \equiv -r_0\kappa_0,$$

and ρ and σ are factors of proportionality. From these,* we get

$$\rho\sigma(u'x') \equiv (U - d^2 x_4^2)\psi(ux),$$

$$\rho\sigma(u'dx') \equiv (U - d^2 x_4)\psi(udx),$$

(udx) being the linear differential expression $u_1 dx_1 + u_2 dx_2 + u_3 dx_3 + u_4 dx_4$; these identities show the transformation to be a contact transformation. Assuming $\rho = \sigma = \psi$, we may readily verify the involutory property. If $d = 0$, the transformation is a point transformation, the usual spherical inversion.

Without entering into details, we may state the following theorems, it being understood that the complex of all points Ω inverts by (A_0) into the complex A.†

The ∞^1 minimum surface elements along a minimum line intersecting the fundamental sphere in P give the same element (P, E), whose plane E belongs to A.

* In the equations just written, the coördinates x, u were chosen rather than the customary x, y, z, p, q, for homogeneity and to emphasize *orientation*.

† The important thing is this: The ∞^2 minimum planes constitute the linear congruence common to the complexes of all points and all planes. By (A_0), this congruence becomes the congruence (special) common to A and the *special* complex whose fundamental sphere is $U = d^2 x_4^2$.

An element of the imaginary circle at infinity gives the ∞^2 elements of a sphere of A, belonging to $\psi = 0$; this sphere is of radius d and its center is on the asymptotic cone of the fundamental sphere.

An element of the sphere $U = d^2x_4^2$ transforms into ∞^1 parallel elements with common point on the imaginary circle at infinity.

The sphere $U = d^2x_4^2$ and the imaginary circle at infinity are "fundamental configurations" for the transformation.

Consider, now, the curve of intersection c of the fundamental sphere and the non-singular minimum developable circumscribing the primitive. The ∞^1 elements common to this developable and the primitive invert into elements of the derived surface along the curve c. Since the planes of the elements belong to A, we have the

THEOREM.—*The intersection of the fundamental sphere and the non-singular minimum developable circumscribing the primitive is a line of curvature on the derived surface along which the principal spheres have four-point contact.*

The double line of theorem 3, § 4, and the curve just mentioned constitute the entire intersection of the derived surface and the fundamental sphere. It is otherwise obvious that any line common to the fundamental sphere and the derived surface which is not a multiple line on that surface must be a line of curvature, since its elements must remain unchanged under the inversion (A).

The theorem enunciated is merely a case of the well-known general theorem:

The ∞^1 surface elements of the primitive surface along a line of curvature invert into ∞^1 elements of the derived surface along a line of curvature.

For the curve of contact of the primitive surface with the circumscribed minimum developable is a singular line of curvature. *

Let the elements (P, E) and (P', E') be such that P and P' lie on a minimum line while E and E' intersect in that line. Then the transformed elements will have the same property, whence the theorem:

The two continuous systems of oriented elements of the primitive along a minimum line in that surface invert into elements of the derived surface along two distinct minimum lines intersecting on the fundamental sphere and lying in a plane belonging to A.

§ 6. *The surface of singularities of the general quadratic spherical complex.*

Particular interest attaches to the case when the primitive surface f of the preceding sections is a cyclide, i. e., a surface of the fourth order containing the imaginary circle at infinity as double line. This surface is sibi-reciprocal under

* Cf. DARBOUX, *Sur une classe remarquable* etc., page 11. Also SNYDER, American Journal of Mathematics, vol. 22, p. 97, 1900.

five commutative spherical inversions, and being non-oriented, under (Ω) as well. The class of the cyclide is 12, and the non-singular minimum developable is of order 16. * By a general inversion in a complex having no special relation to the cyclide this surface is transformed into the surface of singularities of the general quadratic spherical complex. †

We have then the following theorems for this surface:

The surface of singularities of the general quadratic spherical complex is sibi-reciprocal under six commutative general inversions (A_i). *With each* (A_i) *is associated a cyclide* f_i *into which the surface transforms by an inversion* (A_i') *concentric with* (A_i).

The surface is of order and class 24, *and contains the imaginary circle at infinity as twelve-fold line, and also six double lines, the intersections of* f_i *and the fundamental sphere* σ_i *of the corresponding inversion* (A_i). *These double lines are sphero-quartic curves.*

The surface is the envelope in six ways of ∞^2 *spheres belonging to an* A_i *whose centers lie on a Kummer quartic* F_i. ‡ *The six quartics* F_i *are confocal, and are inscribed in the minimum developable of the surface of singularities, which is also the singular minimum developable for that surface. Thus the focal lines of the* F_i *are the focal lines of the surface.*

The surface is enveloped by six systems of bitangent planes each belonging to a developable of order 8 *circumscribed about a sphere concentric with* (A_i) *along a sphero-quartic. The total intersection of the surface with a fundamental sphere* σ_i *is made up of the double line already mentioned, and a curve of order* 16, *which is a line of curvature of the surface along which the principal spheres have four-point contact. The lines of curvature of the surface are algebraic and of order* 32, *and are inverted by each of the six inversions* (A_i') *into the lines of curvature of a cyclide* f_i. *The surface contains* 32 *minimum lines, arranged in six ways in* 16 *pairs, such that the intersection of each pair is on a fundamental sphere* σ_i, *while the plane of each pair belongs to* (A_i).

Since a cyclide f_i depends upon 13, an inversion (A_i) upon 5 essential constants, we have, finally,

The surface of singularities of the general quadratic spherical complex depends upon 18 *essential constants.*

These results are believed to be new.§

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* Cf. DARBOUX, *Sur une classe remarquable de courbes et de surfaces*, p. 305.

† LIE, loc. cit., p. 178. Cf. also SNYDER, Bulletin of the American Mathematical Society, vol. 4, p. 152.

‡ i. e., a quartic containing a double conic.

§ Cf., however, SNYDER, Bulletin of the American Mathematical Society, May, 1900.

§ 7. *Cyclide of Dupin.*

To illustrate the preceding theorems when the surface is annular, we may consider the case when the *primitive is a right line* (or circle). Then the derived surface is enveloped by ∞^1 spheres belonging to the complex A, whose centers lie upon a conic, i. e., is a cyclide of Dupin, of order and class 4, containing the imaginary circle two-fold. The two pencils of oriented planes containing the primitive invert into the spheres of the second generation.

Space does not permit us to carry the discussion further. This relation of the cyclide of Dupin to the right line is believed to be new, and the discussion of the properties of the surface from this point of view is of great simplicity and interest.

§ 8. *Cases of exception to the general theorems.*

In the statement of theorem 6, § 4, the assumption was made that the primitive surface f was not sibi-reciprocal under the spherical inversion (I'). We easily derive the following results, if $f(I')f$.

*If $f(I')f$, the derived surface F is the complete parallel of a surface F', which is sibi-reciprocal under the spherical inversion (I), concentric with (I').** If f_0 be the locus of centers of spheres belonging to (I') and enveloping f, then f_0 and the locus of centers for F are polar reciprocals under a polar reciprocation in the center of inversion with power $\frac{1}{2}s$, s having the same signification as above (§ 4). Theorems 3, 4, 5, 7 (§ 4) still hold.

In particular, if the primitive is a cyclide, then the derived surface is the complete parallel of a second cyclide.

Consider next the case where *the surface is enveloped by ∞^2 spheres belonging to an orthogonal complex*, i. e., is a surface of the type discussed by MOUTARD. We are led to the following:

A non-oriented surface enveloped by ∞^2 spheres of an orthogonal complex I, by the inversion (A_0) which carries (I) over into the complex of all points, inverts into a non-oriented surface which is also enveloped by ∞^2 spheres belonging to (I). The distinction between primitive and derived disappears. The loci of centers are polar reciprocals under polar reciprocation in the fundamental sphere. The intersection of each surface with the fundamental sphere is a focal line of the other surface.* Theorems 4 and 7 still hold.

In particular, if one surface is a cyclide so is the other, either having the above relations to each of a system of five cyclides.

Finally, let the surface be enveloped by ∞^2 spheres belonging to a plane complex E.

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* It is understood that the inversions involved are related as follows: $F(A)F$; $f(A_0)F$; $(A)=(A_0\Omega A_0)$; $(AD')=(D'A)$; $(D')=(A_0I'A_0)$; $(A)=(DID)$; $F'(D)F$.

* In formulas (7), § 5, $d_0=-r_0\sqrt{-1}$.

Theorems 1, 3, 4, 7 of § 4 are true in this case.* The remaining theorems are replaced by the following:

The oriented surface F derived by inversion in a plane complex from a primitive surface f of order m, class n, and order of non-singular minimum developable $2N$, is of order $2(m+N)$, class $2n$, passing n times through the imaginary circle. The locus of centers of F is derived from f by changing its ordinates relative to the fundamental plane in a certain constant ratio.

The case in which the primitive is a central quadric is discussed at length in the paper by the author already referred to. An incomplete analytical discussion is given by DARBOUX, loc. cit., vol. II, p. 335.

The theorems given in section 5 are true in this special case. The equations of the transformation as affecting surface elements may be derived from the general case by a simple transformation with passage to a limit.

§ 9. *Generalization for space of n dimensions, in particular, for* $n = 2$.

It is obvious that the discussion and results developed in the preceding pages for space of three dimensions are extended to space of n dimensions by certain trivial verbal and literal changes and additions. Passing over questions of this character, we may turn our attention to the problem in the plane, in which case the general theorems take the following form:

An oriented curve C enveloped by ∞^1 oriented circles belonging to a linear circular complex A may always be derived by a general inversion (A_0) from a primitive curve regarded as a point locus. The order and class of C are equal, and it intersects the line at infinity only in the circular points. Hence all the foci of C are singular, and all its asymptotes are isotropic. The points of intersection of the primitive curve and fundamental circle are nodes on C. A circle orthogonal at once to the fundamental circle and to the primitive at the point P intersects the curve C also orthogonally in the two points of contact of C and the circle into which P inverts under (A_0). Certain tangents (equal in number to the nodes) to a circle concentric with the fundamental circle are bitangents of C.

If the primitive is of order m, class n, passing r times through each circular point, the derived curve C is of order and class $2(2m+n-4r)$, having at each circular point a multiple point of order $2m+n-4r$. In addition, C has $2(m-r)$ nodes on the fundamental circle, and $2(m-r)$ bitangents touching a concentric circle. The centers of the ∞^1 circles enveloping C lie upon a curve of order $2(m-r)$, class $2m+n-4r$, having at the center of (A_0) a multiple point of order $m-2r$. The locus of centers C_0 and the primitive curve have equal deficiencies. The curves C_0 and C have the same foci.

* The complex E transforms into the complex of all points by inversion in a *plane* complex. Cf. SMITH, loc. cit., for further details.

The two continuous systems each of ∞^1 oriented circles osculating the primitive invert in (A_0) into the osculating circles of the curve C. Apart from the nodes, C intersects the fundamental circle in $2(n-2r)$ points, at which the osculating circles have four-point contact.

As an application consider the case when the primitive curve c is a bicircular quartic, and, therefore, of order 4, class 8, passing twice through each circular point. Such a curve is sibi-reciprocal under four commutative inversions and also under (Ω). One has the following results:

The curve C is of order and class 16, passes 8 times through each circular point, and is sibi-reciprocal under five commutative inversions (A_i). By inversions concentric with the (A_i) C inverts into 5 bicircular quartics c_i. The curve C is enveloped in five ways by ∞^1 oriented circles belonging to an A_i whose centers lie upon a quartic curve of deficiency unity. These five quartic curves are confocal, their foci being the foci of C. There are 20 double points on C, four each on a fundamental circle, the points of intersection namely of c_i and that circle. Also C is touched by 20 double tangents, arranged in five sets of four each, touching circles concentric with the fundamental circles.

The curve C just discussed is well known as the curve of singularities of the general quadratic circular complex. Its equation contains 12 essential parameters, to be accounted for by the dependence of a bicircular quartic upon 8 and of a general inversion upon 4. The properties just given are believed to be new.*

SHEFFIELD SCIENTIFIC SCHOOL,
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*Cf. again SNYDER, Bulletin of the American Mathematical Society, May, 1900.

ON CERTAIN RELATIONS AMONG THE THETA CONSTANTS*

BY

J. I. HUTCHINSON

In the case of the theta functions of two variables ($p = 2$), the theta constants (i. e., the ten functions with zero arguments) are well known to be expressible in terms of any four ($2^p = 2^2 = 4$) of them, the most interesting relations being those in which the characteristics of the four form a GÖPEL quadruple. Thus, in the formulæ given by KRAUSE in §11 of his book, *Die Transformation der hyperelliptischen Funktionen erster Ordnung*, we have certain combinations of the constants rationally expressed in terms of ϑ_5, ϑ_0, ϑ_{12}, ϑ_{34} (formulæ (1)), while each constant is expressed irrationally in terms of the same (formulæ (3)).

In the following pages the corresponding results are given for the case of the general theta functions ($p = p$): viz., formulæ (I), (II) below are a direct generalization of formulæ (1) of KRAUSE just referred to, and (IV) is a generalization of (3), while (III) is a new relation not existing for the case $p = 2$.†

I use the definitions and notations of FROBENIUS (*Das Additionstheorem der Thetafunctionen*, Crelle's Journal, vol. 89, 1880) and start with the formula (loc. cit., p. 201):

$$\sum_{a=0}^{r-1}(P_a)\,\vartheta[A_a]\,(u+v)\,(u-v)\,(a+b)\,(a-b) = \sum_{a}(P_a)\,\vartheta[A_a]\,(u+b)\,(u-b)\,(a+v)\,(a-v). \tag{1}$$

In this formula $A_0 = A$, $A_1, \cdots, A_{r-1}$ are the $r = 2^p$ characteristics of a GÖPEL system, and $P_a = AA_a$.

In (1) replace u and b by $u + C$ and $b + C$, where C denotes any system of half periods. The formula becomes:

$$\sum(P_aC)\,\vartheta[A_aC]\,(u+v)\,(u-v)\,(a+b)\,(a-b) = (C)\sum(P_aC)\,\vartheta[A_a]\,(u+b)\,(u-b)\,(a+v)\,(a-v). \tag{2}$$

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† Special examples of these relations for the case $p = 3$ were communicated to me by Mr. H. W. KUHN and suggested the generalizations here made.

Putting $u = v = a = b = 0$ we have

$$\text{(I)} \qquad \sum (P_\alpha C)\, \vartheta^4[A_\alpha C] = (C) \sum (P_\alpha C)\, \vartheta^4[A_\alpha] .$$

In (2) replace u by $u + P_\beta$. We obtain

$$\text{(3)} \quad \begin{aligned} &\sum (P_\alpha C) \binom{P_\beta}{A_\alpha C} \vartheta[A_\alpha P_\beta C]\,(u+v)\,(u-v)\,\vartheta[A_\alpha C]\,(a+b)\,(a-b) \\ &= (C) \sum (P_\alpha C) \binom{P_\beta}{A_\alpha} \vartheta[A_\alpha P_\beta]\,(u+b)\,(u-b)\,\vartheta[A_\alpha]\,(a+v)\,(a-v) . \end{aligned}$$

Making the arguments zero we have

$$\text{(4)} \quad \begin{aligned} &\sum (P_\alpha C) \binom{P_\beta}{A_\alpha C} \vartheta^2[A_\alpha P_\beta C]\ \vartheta^2[A_\alpha C] \\ &\qquad = (C) \sum (P_\alpha C) \binom{P_\beta}{A_\alpha} \vartheta^2[A_\alpha P_\beta]\ \vartheta^2[A_\alpha] . \end{aligned}$$

This formula can be simplified. Suppose the GÖPEL system to be represented in the form

$$AB_i , \quad A_\beta B_i \qquad (i = 0, 1, \cdots, \tfrac{1}{2}r - 1)$$

where B_0 is the characteristic zero. Then (4) can be written:

$$\begin{aligned} &[1 + (P_\beta , C)] \sum_i (B_i C) \binom{P_\beta}{AB_i C} \vartheta^2[AB_i C]\, \vartheta^2[A_\beta B_i C] \\ &\qquad = (C)\,[1 + (P_\beta , C)] \sum_i (B_i C) \binom{P_\beta}{AB_i} \vartheta^2[AB_i]\ \vartheta^2[A_\beta B_i] . \end{aligned}$$

In reducing to this form we notice that the B_i are contained among the P_α and hence $(P_\beta , B_i) = 1$.

The formula so obtained vanishes identically if $(P_\beta , C) = -1$. If $(P_\beta , C) = 1$ we have

$$\text{(II)} \quad \begin{aligned} &\sum (B_i C) \binom{P_\beta}{AB_i C} \vartheta^2[AB_i C]\, \vartheta^2[A_\beta B_i C] \\ &\qquad = (C) \sum (B_i C) \binom{P_\beta}{AB_i} \vartheta^2[AB_i]\ \vartheta^2[A_\beta B_i] . \end{aligned}$$

It is to be observed in this connection that

$$(AB_i)\,(A_\beta B_i) = (A)\,(A_\beta)\,(P_\beta , B_i) = (A)\,(A_\beta) ,$$

and

$$(AB_i C)\,(A_\beta B_i C) = (A)\,(A_\beta)\,(P_\beta , C) = (A)\,(A_\beta) .$$

Consequently if A and A_β are of opposite character (i. e., one odd and the other even), formula (II) vanishes identically.

Again, replacing u, v, a, b by $u+\frac{1}{2}P_\gamma$, $v+\frac{1}{2}P_\gamma$, $a+\frac{1}{2}P_\gamma$, $b+\frac{1}{2}P_\gamma$ in formula (3), we have

$$\sum(P_\alpha C)\begin{pmatrix}P_\beta P_\gamma\\ A_\alpha C\end{pmatrix}\vartheta[A_\alpha C](a-b)\,\vartheta[A_\alpha P_\beta C](u-v)\,\vartheta[A_\alpha P_\gamma C](a+b)\cdot\vartheta[A_\alpha P_\beta P_\gamma C](u+v)$$
$$=(C)\sum(P_\alpha C)\begin{pmatrix}P_\beta P_\gamma\\ A_\alpha\end{pmatrix}\vartheta[A_\alpha](a-v)\,\vartheta[A_\alpha P_\beta](u-b)\,\vartheta[A_\alpha P_\gamma](a+v)\cdot\vartheta[A_\alpha P_\beta P_\gamma](u+b).$$

Making the arguments zero we have

$$\sum(P_\alpha C)\begin{pmatrix}P_\beta P_\gamma\\ A_\alpha C\end{pmatrix}\vartheta[A_\alpha C]\,\vartheta[A_\alpha P_\beta C]\,\vartheta[A_\alpha P_\gamma C]\,\vartheta[A_\alpha P_\beta P_\gamma C]$$
$$=(C)\sum(P_\alpha C)\begin{pmatrix}P_\beta P_\gamma\\ A_\alpha\end{pmatrix}\vartheta[A_\alpha]\,\vartheta[A_\alpha P_\beta]\,\vartheta[A_\alpha P_\gamma]\,\vartheta[A_\alpha P_\beta P_\gamma]. \tag{5}$$

This formula can be simplified by representing the characteristics of the GÖPEL system in the form:

$$AD_j,\quad A_\beta D_j,\quad A_\gamma D_j,\quad AA_\beta A_\gamma D_j \qquad (j=0,\,1,\,\cdots,\,\tfrac{1}{4}r-1).$$

Equation (5) becomes:

$$[1+(P_\beta,\,C)][1+(P_\gamma,\,C)]\sum(D_jC)\begin{pmatrix}D_j\\ AA_\beta A_\gamma\end{pmatrix}\vartheta[AD_jC]\cdot\vartheta[A_\beta D_jC]\,\vartheta[A_\gamma D_jC]\,\vartheta[AA_\beta A_\gamma D_jC]$$
$$=[1+(P_\beta,\,C)][1+(P_\gamma,\,C)](C)\begin{pmatrix}C\\ A\end{pmatrix}\begin{pmatrix}A_\beta A_\gamma\\ C\end{pmatrix}\sum(D_jC)\begin{pmatrix}D_j\\ AA_\beta A_\gamma\end{pmatrix}\cdot\vartheta[AD_j]\,\vartheta[A_\beta D_j]\,\vartheta[A_\gamma D_j]\,\vartheta[AA_\beta A_\gamma D_j]. \tag{6}$$

If either $(P_\beta,\,C)$ or $(P_\gamma,\,C)$ is equal to -1, formula (6) vanishes identically. If $(P_\beta,\,C)=(P_\gamma,\,C)=1$ we have

$$\sum_j(D_jC)\begin{pmatrix}D_j\\ AA_\beta A_\gamma\end{pmatrix}\vartheta[AD_jC]\,\vartheta[A_\beta D_jC]\,\vartheta[A_\gamma D_jC]\,\vartheta[AA_\beta A_\gamma D_jC]$$
$$=(C)\begin{pmatrix}C\\ A\end{pmatrix}\begin{pmatrix}A_\beta A_\gamma\\ C\end{pmatrix}\sum_j(D_jC)\begin{pmatrix}D_j\\ AA_\beta A_\gamma\end{pmatrix}\vartheta[AD_j]\,\vartheta[A_\beta D_j]\,\vartheta[A_\gamma D_j]\cdot\vartheta[AA_\beta A_\gamma D_j]. \tag{III}$$

Further results are most easily obtained by starting with the formula:

$$(7)\quad \begin{aligned}&[\sum_a \sqrt{P_a}\,\vartheta[A_a](u+v)(u-v)]\,[\sum_a \sqrt{P_a}\,\vartheta[A_a](a+b)(a-b)]\\ &\quad = [\sum_a \sqrt{P_a}\,\vartheta[A_a](u+b)(u-b)]\,[\sum \sqrt{P_a}\,\vartheta[A_a](a+v)(a-v)]\,,\end{aligned}$$

which is given in the memoir cited above (p. 200). We assume here that the characteristics A_a are all even. Replace u, b by $u+C$, $b+C$. We obtain:

$$\begin{aligned}&\Big[\Sigma\binom{C}{A_a}\sqrt{P_a}\vartheta[A_aC](u+v)(u-v)\Big]\Big[\Sigma\binom{C}{A_a}\sqrt{P_a}\vartheta[A_aC](b+a)(b-a)\Big]\\ &\quad = [\textstyle\sum(A_a, C)\sqrt{P_a}\vartheta[A_a](u+b)(u-b)]\,[\sum\sqrt{P_a}\vartheta[A_a](a+v)(a-v)]\,.\end{aligned}$$

Making the arguments zero and extracting the root we have

$$\Sigma\binom{C}{A_a}\sqrt{P_a}\vartheta^2[A_aC] = \sqrt{\textstyle\sum(A_a, C)\sqrt{P_a}\vartheta^2[A_a]\cdot\sum\sqrt{P_a}\vartheta^2[A_a]}\,.$$

Multiply both members by $\sqrt{P_\beta}$, and add all the 2^p equations obtained by giving all possible combinations of signs to the radicals $\sqrt{P_\alpha}$, $\sqrt{P_\beta}$. Using $\mathbf{S}$ to denote this summation and observing that $\mathbf{S}\sqrt{P_\alpha}\sqrt{P_\beta} = 2^p(P_a)$, or 0 according as a is or is not equal to β, we have

$$(\mathrm{IV})\quad 2^p\binom{C}{A_a}(P_a)\vartheta^2[A_aC] = \mathbf{S}\sqrt{P_\alpha}\sqrt{\textstyle\sum(A_\alpha, C)\sqrt{P_\alpha}\vartheta^2[A_\alpha]\cdot\sum\sqrt{P_\alpha}\vartheta^2[A_\alpha]}\,.$$

This formula vanishes identically if $(A_aC) = -1$.

The relation (IV) enables one to express all of the $2^{p-1}(2^p+1)$ theta constants homogeneously and algebraically in terms of 2^p of them, viz., the $\vartheta[A_a]$. Since the thetas depend on $\frac{1}{2}p(p+1)+1$ parameters (including the common multiplicative constant) it is evident that for $p>2$ formula (IV) gives a parametric representation of the theta constants with $2^p - \frac{1}{2}p(p+1) - 1$ superfluous parameters; or, in other words, the 2^p parameters $\vartheta[A_a]$ satisfy this number of relations. These relations would seem to follow at once from formula (3), p. 204 of the memoir of FROBENIUS already cited when the arguments u are made to vanish.

CORNELL UNIVERSITY,
December, 1899.

ON THE GROUPS WHICH HAVE THE SAME GROUP OF ISOMORPHISMS*

BY

G. A. MILLER

The main object of this paper is the determination of all the possible groups whose group of isomorphisms is either the symmetric group of order 6 or the symmetric group of order 24. We shall also determine the infinite system of groups whose group of cogredient isomorphisms is the former of these two symmetric groups. It will be proved that this system includes one and only one group (which is not the direct product of an abelian and a non-abelian group) for every power of 2.

It is well known that every simple isomorphism of a group G with itself may be obtained by transforming G by means of operators that transform it into itself.† In what follows we shall generally employ this method of making G simply isomorphic with itself.

In a few cases it will be convenient to employ two special methods, which we proceed to explain.

The first of these two methods may be employed when G contains a subgroup H' which is composed entirely of operators which are selfconjugate under G and which is also simply isomorphic to a quotient group of G with respect to a selfconjugate subgroup which includes H'. In this case we may evidently multiply all of the operators of each one of the various divisions of G with respect to this quotient group by the corresponding operator of H' and thus obtain a simple isomorphism of G with itself.—To illustrate this method we may employ the direct product G_{12} of the symmetric group of order 6 and an operator s_1 of order two. If we multiply each of the six operators of G_{12} which are not contained in its cyclical subgroup of order 6 by s_1 we obtain a simple isomorphism of G_{12} with itself. It is evident that this isomorphism corresponds to the selfconjugate operator of order two in the group of isomorphisms of G_{12}.‡—It is important to observe that any operator t_1 of the group of isomorphisms of G which is obtained in this manner is selfconjugate under this group of isomorphisms whenever H' is composed of characteristic operators

* Presented to the Society December 28, 1899. Received for publication January 18, 1900.

† FROBENIUS: Berliner Sitzungsberichte, 1895, p. 184.

‡ The group of isomorphisms of G_{12} is simply isomorphic to G_{12}.

of G and each of the divisions of G which is multiplied by the same operator of H' corresponds to itself in every simple isomorphism of G with itself. To prove this we let s_a and t_2 represent any operator of G and its group of isomorphisms respectively. If $t_2^{-1}s_at_2 = s_b$ and $t_1^{-1}s_bt_1 = h_1s_b$ then $(t_2^{-1}t_1t_2)^{-1}s_bt_2^{-1}t_1t_2 = h_1s_b$; i. e., $t_2^{-1}t_1t_2 = t_1$.*

The second of these special methods of making G simply isomorphic with itself can be employed when G is an abelian group which contains operators whose order exceeds two. In this case we may write after each operator of G its a power, where a is any integer prime to the order of G†, and thus obtain a simple isomorphism of G with itself. It is easy to prove that such an isomorphism of G with itself must correspond to a selfconjugate operator t_3 in its group of isomorphisms. For if $t_2^{-1}s_at_2 = s_b$ and $t_3^{-1}s_bt_3 = s_b^a$ then

$$(t_2t_3t_2^{-1})^{-1}s_at_2t_3t_2^{-1} = s_a^a,$$

i. e., $t_2t_3t_2^{-1} = t_3$. Hence we have

THEOREM I. *If an abelian group contains operators whose order exceeds two its group of isomorphisms contains selfconjugate operators.*‡

We proceed to give a few other elementary theorems (II ··· VI) in the form in which we shall need them in what follows. Theorem V will not be directly employed but it may be useful for the purpose of arriving at some of the following results by different methods.

THEOREM II. *If a group G is generated by two characteristic subgroups which have only identity in common its group of isomorphisms is the direct product of the groups of isomorphisms of these two characteristic subgroups.*

Any isomorphism of G with itself is completely determined by the operators which correspond to any set of generators of G. Hence the isomorphisms of the given characteristic subgroups with themselves will completely determine the isomorphism of G with itself. Since G is the direct product of these subgroups their isomorphisms are entirely independent of each other and hence the group of isomorphisms of G is the direct product of their groups of isomorphisms.

COROLLARY. *If the order of an abelian group is $p_1^{a_1}p_2^{a_2}p_3^{a_3}\cdots$ ($p_1, p_2, p_3, \cdots$ being different prime numbers) its group of isomorphisms is the direct product of the groups of isomorphisms of its subgroups of orders $p_1^{a_1}$, $p_2^{a_2}$, $p_3^{a_3}, \cdots$.*

* The operators of the group of isomorphisms of G may be supposed to transform G into itself and to be fully determined by the way in which they transform the operators of G. Cf. BURNSIDE, *Theory of Groups of a Finite Order*, 1897, p. 226; also MILLER, Bulletin of the American Mathematical Society, vol. 5, 1899, p. 245.

† The order of G diminished by unity is evidently a suitable value for a.

‡ It is well known that the group of isomorphisms of a group which does not contain any operator whose order exceeds 2 is simple except in case the order of the group is 4.

THEOREM III. *The group of isomorphisms of an abelian group A is abelian when A is cyclical and it is non-abelian when A is non-cyclical.*

When A is cyclical its group of isomorphisms is simply isomorphic to the operators which transform a generator of A into its various powers which are prime to its order. Hence this group of isomorphisms is abelian. When A is non-cyclical it contains a non-cyclical group P whose order is a power of a single prime number. Since P is generated by its operators of highest order* its group of isomorphisms is simply isomorphic to a transitive substitution group whose degree is equal to the number of these operators of highest order. As the order of this transitive group clearly exceeds its degree it is non-abelian. Hence the group of isomorphisms of A must also be non-abelian.

THEOREM IV. *The necessary and sufficient condition that a cyclical group of order n is the group of isomorphisms of some group is that n is of the form $p^\alpha(p-1)$, p being an odd prime number.*

The group of isomorphisms of a non-abelian group cannot be cyclical since its group of cogredient isomorphisms cannot be cyclical.† Hence this theorem follows directly from the preceding theorem and the well known forms of numbers which have primitive roots. It may be observed that a cyclical group of an odd order cannot be a group of isomorphisms and that there is no group which has a cyclical group of order $8, 14, 24, 26, \cdots$ as a group of isomorphisms.

THEOREM V. *If the order of an abelian group A is divisible by an odd prime number p its group of isomorphisms contains an operator of order $p-1$.*

Let p^α be the highest power of p that divides the order of A. It follows from theorem II that we may confine our attention to the group P of order p^α which is contained in A. Since P is abelian a simple isomorphism of P with itself may be obtained by writing after each one of its operators the a power of this operator, a being any number prime to p. If p^β is the highest order of an operator of P and if a is so chosen that it belongs to exponent $p-1$ modulo p^β, the given simple isomorphism of P with itself will clearly correspond to an operator of order $p-1$ in its group of isomorphisms.

THEOREM VI. *If an operator A_1 of the group of cogredient isomorphisms G_1 of the group G is of order p^α (p being any prime number) and if some operator of G_1 transforms A_1 into its k power, then at least one operator of order p^α corresponds to A_1 in the isomorphism between G and G_1 whenever*

* It is easy to see that any abelian group is generated by its operators of highest order and that its group of isomorphisms can be represented as a transitive substitution group whose degree is equal to the number of these operators of highest order.

† Comptes Rendus, 1899, vol. 128, p. 229; cf. FITE, Bulletin of the American Mathematical Society, 1899, vol. 6, p. 11.

$k \not\equiv 1 \bmod. p$; when $k \equiv 1 \bmod. p^\gamma$ but $\not\equiv 1 \bmod. p^{\gamma+1}$ then at least one operator of order $p^{\alpha+\gamma'}$ $(\gamma' \geqq \gamma)$ corresponds to A_1 in the given isomorphism between G and G_1.

Let H be the subgroup of G which corresponds to identity in the given isomorphism between G and G_1 and let p^t be the highest power of p which divides the order of H. To the subgroup of G_1 which is generated by $A_1^{p^{\alpha-1}}$ there corresponds an abelian group which includes a group P of order p^{t+1}. Let S be an operator of P of the lowest order which corresponds to $A_1^{p^{\alpha-1}}$. The operator S may evidently be used as one of the independent generators of P. Since some operators of G_1 transform $A_1^{p^{\alpha-1}}$ into its kth power the corresponding operators of G must transform S into its k power multiplied by some operator of H. The p power of this transform must equal S^p since each of the operators of H is commutative to every operator of G. As this is clearly impossible unless $S^p = 1$ the first part of the theorem is proved.*

To prove the second part we observe that an operator A_2 (contained in G_1) of order $p^{\gamma+1}$ is transformed into its $p^\gamma + 1$ power by some operators of G_1. Any one of the corresponding operators of G must therefore be transformed into its $p^\gamma + 1$ power multiplied by some operator of H. As before we may select one of these operators S_1 of lowest order as one of the independent generators of (P_1) the subgroup of order $p^{t+\gamma+1}$ which corresponds to the group generated by A_2. The $p^{\gamma+1}$ power of the given transform of S_1 must equal $S_1^{p^{\gamma+1}}$; i. e., the $p^\gamma + 1$ power of S_1 must have the same $p^{\gamma+1}$ power as S_1 has, since S_1 is an independent generator of P_1. From this it follows that the order of $S_1^{p^\gamma}$ cannot exceed $p^{\gamma+1}$ and hence the order of S_1 cannot exceed $p^{2\gamma+1}$. This completes the proof of the theorem.

By means of these theorems it is easy to determine all the groups whose group of isomorphisms is the symmetric group of order six (G_6). If such a group K is abelian it cannot contain any operator whose order exceeds two, according to theorem I. Hence the four-group is the only abelian group whose group of isomorphisms is G_6. If K is non-abelian its group of cogredient isomorphisms must be G_6 since a group of cogredient isomorphisms cannot be cyclical.† We shall therefore first consider the more general problem to determine all the possible groups whose group of cogredient isomorphisms is G_6. This problem is of considerable interest in itself, especially since we arrive at a very simple infinite system of groups.

Let K' be any group whose group of cogredient isomorphisms is G_6 and let H represent the subgroup of K' which corresponds to identity in the isomorphism between K' and G_6. The order of each operator of K' which corresponds to an operator of order three in G_6 must be divisible by three. From theorem

*Cf. HÖLDER, Mathematische Annalen, 1895, vol. 46, p. 351.

†Comptes Rendus, 1889, vol. 128, p. 229.

VI we observe that at least two of these operators (s, s^2) of K' are of order three. Let t be an operator of K' which corresponds to an operator of order two in G_6 and suppose that t is so chosen that its order 2^a is the smallest possible power of 2. The operators t and s generate a selfconjugate subgroup of K'.* We may suppose that this selfconjugate subgroup contains only one subgroup of order 3, since the number of operators of order 3 which correspond to an operator of order 3 in G_6 is 3^δ and hence at least one of the 3^δ subgroups which they generate must be transformed into itself by t.

Let 2^l be the highest power of 2 which divides the order of H and let A' represent the subgroup of order 2^l which is contained in H. Then A' and t generate an abelian group of order 2^{l+1} whose independent generators may be so chosen as to include t. Hence t^2 may be used as one of the independent generators of H and K' is the direct product of the group generated by s and t, and of the subgroup of H which is generated by all of its independent generators except t^2.

The operators s and t^2 generate a cyclical group whose generators are transformed by t either into their $2^{a-1}+1$ or into their 2^a+1 power. One of the two numbers $2^{a-1}+1$, 2^a+1 is congruent to 0 mod. 3 while the other is congruent to 2 mod. 3, t must, of course, transform the given generators into the power which is congruent to 2 mod. 3. As there is only one such group for every value of a we have proved

THEOREM VII. *If a group has the symmetric group of order six for its group of cogredient isomorphisms and is not the direct product of two subgroups its order is* 3.2^a *and it contains three cyclical subgroups of order* 2^a *and only one subgroup of order* 3. *For every value of* $a > 0$ *there is one and only one such group. Every other group which has the symmetric group of order six for its group of cogredient isomorphisms is the direct product of one of the given groups of order* 3.2^a *and some abelian group.*

It is now easy to see that G_6 is the only non-abelian group whose group of isomorphisms is G_6. For if such a group K were the direct product of two subgroups one of these subgroups would have to be abelian while the other would have G_6 for its group of cogredient isomorphisms. Hence we may assume that the latter group would be one of the given groups of order 3.2^a. If a exceeds 1 such a group contains a characteristic operator of order 2 and a subgroup which includes this operator of order 2 and whose order is one-half the order of the group. Hence the group of isomorphisms of such a group must contain a selfconjugate operator of order 2 whenever $a > 1$. The group K must

* The general theorem may be stated as follows: If a subgroup contains at least one operator which corresponds to each operator of the group of cogredient isomorphisms of the entire group this subgroup is selfconjugate.

therefore be either G_6 or the direct product of G_6 and some abelian group. We have already seen that this abelian group could not be of order 2 and it follows directly from theorem II that it could not be of a larger order. Hence G_6 is the only non-abelian group whose group of isomorphism is G_6.

Having determined the groups whose group of isomorphisms is the symmetric group of order six we proceed to consider all the possible groups whose group of isomorphisms is the symmetric group of order 24. We shall denote this group by G_{24}. It follows almost directly from theorem I that such a group L cannot be abelian. If it is non-abelian its group of cogredient isomorphisms must be the four-group, the alternating group of order 12, or G_{24} itself. We shall consider these three cases in order.

If the group of cogredient isomorphisms of L is the four-group the selfconjugate operators of L constitute a characteristic subgroup with respect to which L is isomorphic to the four-group. Any operator which transforms L into a contragredient isomorphism with itself must permute some of the divisions that correspond to the three operators of order two in the given quotient group. Hence L cannot admit any except cogredient isomorphisms with itself in which each of these divisions corresponds to itself. In particular each of the selfconjugate operators of L must correspond to itself in all such isomorphisms. From this it is not difficult to see that the order of L must be 8 and it is well known that the quaternion group is the only group of order 8 whose group of isomorphisms is G_{24}. *The quaternion group is therefore the only group which has the four-group for its group of cogredient isomorphisms and also G_{24} for its group of isomorphisms.*

There are three well-known groups which have the alternating group of order 12 *for their group of cogredient isomorphisms and G_{24} for their group of isomorphisms; viz., the alternating group of order* 12, *the direct product of this alternating group and an operator of order two,* and the group of order* 24 *which has no subgroup of order* 12.† We proceed to prove that *there is no other group which has this property.* Since the groups of orders 12 and 24 are well known we may confine our attention to the groups of higher orders.

Since the group of cogredient isomorphisms of L includes all its operators of order 3, L cannot contain any selfconjugate operator of this order. Hence its order must be $2^a \cdot 3$. It remains to prove $a \leqq 3$. If $a > 3$ the subgroup of order 2^a which is included in L cannot contain three cyclical subgroups of order 2^{a-1}. This is evident when the subgroup of order 2^a is abelian. If it is non-abelian suppose that s_1 and s_2 $(s_1^2 = s_2^2)$ generate two subgroups of order 2^{a-1}. From the conditions that $s_1^2 = s_2^2$ and that s_1, s_2 are not commutative it follows that $s_2^{-1} s_1 s_2 = s_1^{2^{a-2}+1}$ and $(s_1 s_2)^2 = s_1 s_2^2 s_2^{-1} s_1 s_2 = s_1^{2^{a-2}+4}$. Hence the order of

*Quarterly Journal of Mathematics, 1898, vol. 30, p. 258.

†Ibid., 1896, vol. 28, p. 274.

$s_1 s_2$ cannot exceed 2^{a-2} when $a > 3$. Since the squares of all the operators of an abelian group of order 2^{a-1} constitute a subgroup of order $2^{a-\beta-1}$, where β is the number of its independent generators, and since each of the three conjugate subgroups of order 2^{a-1} which are contained in L has at least two independent generators, the squares of all the operators of L must constitute a subgroup whose order cannot exceed 2^{a-3}.

It is now easy to see that L admits contragredient isomorphisms with itself which do not permute any of its divisions which correspond to the four-group in its group of cogredient isomorphisms whenever $a > 3$. Hence the three given groups are the only groups whose group of isomorphisms is G_{24} and whose group of cogredient isomorphisms is the alternating group of order 12.

It remains only to determine the groups which have G_{24} both for their group of isomorphisms and also for their group of cogredient isomorphisms. Such a group L must be isomorphic to G_{24} with respect to its selfconjugate operators. From theorems VI and II it follows that it must contain operators of order 3 and that its order cannot be divisible by 9. If L were to contain a selfconjugate operator of order 2 it would have a contragredient isomorphism with itself since G_{24} has a selfconjugate subgroup of order 12. Hence G_{24} *is the only group whose group of isomorphisms and cogredient isomorphisms is* G_{24}.

We have now examined all the possible cases and found that *there are just five groups whose group of isomorphisms is* G_{24}; three of these are of order 24 one is of order 12, and one is of order 8.

CORNELL UNIVERSITY,
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DIE HESSISCHE UND DIE CAYLEY'SCHE CURVE*

VON

PAUL GORDAN

Herr WHITE hat die Curven untersucht, welche mit einer gegebenen Curve 3ter Ordnung die Hessische Δ und die Cayley'sche s gemein haben. Im Anschluss an diese schöne Arbeit werden hier die Curven

$$\bar{\Delta} \text{ und } \bar{s}$$

aufgestellt, deren Hessische und Cayley'sche Curven gegebene Curven 3ter Ordnung

$$f(x) \text{ resp. } f(u)$$

sind.

Während die Δ und s den f eindeutig entsprechen gilt diess im Allgemeinen keineswegs von den $\bar{\Delta}$ und $\bar{s}$. Je nach der Beschaffenheit von f können verschiedene Fälle eintreten. Entweder entsprechen 3 verschiedene $\bar{\Delta}$, $\bar{s}$, oder eine, oder ganze Schaaren, oder endlich keine solche Curven.

§ 1.

Unterscheidungsmerkmale der Curven 3ter Ordnung.

Die wesentlichsten Unterscheidungsmerkmale für Curven 3ter Ordnung f bieten die Anzahl ihrer Doppelpunkte und Rückkehrpunkte und das Verschwinden ihrer Invarianten. Hat f mehr als einen Doppelpunkt so zerfällt es in Curven niederer Ordnung.

§ 2.

Eintheilung der Curven 3ter Ordnung.

Man kann nach diesen Merkmalen die Curven 3ter Ordnung in 10 Arten eintheilen, welche wir durch

$$C_1, C_2, \cdots, C_{10}$$

bezeichnen und so definiren:

1. Die C_1 haben keinen Doppelpunkt. Die Invariante S verschwindet nicht.
2. Die C_2 haben keinen Doppelpunkt. Die Invariante S verschwindet.
3. Die C_3 haben einen Doppelpunkt.

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4. Die C_4 haben zwei Doppelpunkte. Sie zerfallen in einen Kegelschnitt und eine ihn in 2 Punkten schneidende Gerade.

5. Die C_5 haben 3 Doppelpunkte. Sie zerfallen in 3 Gerade, die ein Dreieck bilden.

6. Die C_6 haben einen Rückkehrpunkt.

7. Die C_7 zerfallen in einen Kegelschnitt und eine ihn berührende Gerade.

8. Die C_8 zerfallen in 3 Gerade, welche sich in einem Punkt schneiden.

9. Die C_9 arten in 2 Gerade aus, deren eine doppelt zu rechnen ist.

10. Die C_{10} arten in 1 Gerade aus, welche dreifach zu rechnen ist.

§ 3.

Die Normalformen der C.

Bei passender Wahl des Coördinatensystems lässt sich die Gleichung einer Curve 3ter Ordnung vereinfachen. Für verschiedene Curven bedarf man dazu verschiedene Coördinatensysteme. Die einfachsten Formen der Gleichungen der C nennen wir ihre Normalformen und wählen als solche die folgenden:

$$C_1 = x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3;$$
$$C_2 = x_1^3 + x_2^3 + x_3^3;$$
$$C_3 = x_2^3 + x_3^3 + 6x_1x_2x_3;$$
$$C_4 = x_1^3 + 6x_1x_2x_3;$$
$$C_5 = 6x_1x_2x_3;$$
$$C_6 = 3x_1x_2^2 + x_3^3;$$
$$C_7 = 3x_1x_2^2 + 3x_1^2x_3;$$
$$C_8 = x_2^3 + x_3^3;$$
$$C_9 = 3x_2^2x_3;$$
$$C_{10} = x_1^3.$$

In den folgenden Paragraphen werden die Werthe der Covarianten

$$\Delta,\ s$$

und der Invarianten

$$S,\ T,\ R$$

dieser Normalformen angegeben und sodann ihre charakteristischen Beziehungen aufgestellt.

§ 4.

Die Hessische Form der Normalformen.

$$\Delta(C_1) = -6m^2(x_1^3 + x_2^3 + x_3^3) + 6(2m^3 + 1)x_1x_2x_3;$$
$$\Delta(C_2) = 6x_1x_2x_3;$$

$$\Delta(C_3) = -6(x_2^3 + x_3^3) + 12x_1x_2x_3;$$
$$\Delta(C_4) = -6x_1^3 + 12x_1x_2x_3;$$
$$\Delta(C_5) = 12x_1x_2x_3;$$
$$\Delta(C_6) = -6x_2^2x_3;$$
$$\Delta(C_7) = -6x_1^3;$$
$$\Delta(C_8) = \Delta(C_9) = \Delta(C_{10}) = 0.$$

§ 5.

Die Cayley'sche Form der Normalformen.

$$s(C_1) = -6m(u_1^3 + u_2^3 + u_3^3) + 6(4m^3 - 1)u_1u_2u_3;$$
$$s(C_2) = -6u_1u_2u_3;$$
$$s(C_3) = -6u_1^3 + 24u_1u_2u_3;$$
$$s(C_4) = 24u_1u_2u_3;$$
$$s(C_5) = 24u_1u_2u_3;$$
$$s(C_6) = -6u_1^2u_3;$$
$$s(C_7) = -3u_3^3;$$
$$s(C_8) = s(C_9) = s(C_{10}) = 0.$$

§ 6.

Die Invariante S der Normalformen.

$$S(C_1) = 24m(m^3 - 1);$$
$$S(C_3) = S(C_4) = S(C_5) = 24;$$
$$S(C_2) = S(C_6) = S(C_7) = S(C_8) = S(C_9) = S(C_{10}) = 0.$$

§ 7.

Die Invariante T der Normalformen.

$$T(C_1) = 6(8m^6 + 20m^3 - 1);$$
$$T(C_2) = -6;$$
$$T(C_3) = T(C_4) = T(C_5) = 48;$$
$$T(C_6) = T(C_7) = T(C_8) = T(C_9) = T(C_{10}) = 0.$$

§ 8.

Die Invariante R der Normalformen.

$$R(C_1) = 36(8m^3 + 1)^3;$$
$$R(C_2) = 36;$$
$$R(C_3) = R(C_4) = \cdots = R(C_{10}) = 0.$$

§ 9.

Relationen zur Charakterisirung der Arten C.

Die 10 Arten C_ρ der Curven 3ter Ordnung f sind im § 3 im Wesentlichen geometrisch definirt worden. Wir wollen nunmehr die Relationen hinschreiben-welche zwischen ihren Covarianten und Invarianten bestehen, wenn f einer be, stimmten Art angehört.

1. Für C_1 $\quad R \neq 0\,, \quad S \neq 0\,.$
2. Für C_2 $\quad R \neq 0\,, \quad S = 0\,.$
3. Für C_3 $\quad R = 0\,, \quad \Pi = Ts - St = 0\,.$
4. Für C_4 $\quad \Pi = 0\,, \quad (f\Delta u) \neq 0\,.$
5. Für C_5 $\quad (f\Delta u) = 0\,, \quad S \neq 0\,.$
6. Für C_6 $\quad S = T = 0\,, \quad \mathrm{K} \neq 0\,.$
7. Für C_7 $\quad S = T = \mathrm{K} = 0\,, \quad \Delta \neq 0\,.$
8. Für C_8 $\quad \Delta = 0\,, \quad F = (\Theta\Theta u)^2 \neq 0\,.$
9. Für C_9 $\quad \Delta = F = 0\,, \quad \Theta \neq 0\,.$
10. Für C_{10} $\quad \Theta = 0\,.$

§ 10.

Definition des Normaldreieckes.

Hat man aus diesen Relationen zwischen den Coefficienten der Gleichung der Curve 3ter Ordnung f erkannt, zu welcher Art sie gehört, so handelt es sich darum, sie in die Normalform zu transformiren. Diess geschieht, indem wir ein Dreieck angeben, auf das bezogen sie in der Normalform erscheint. Ein solches Dreiecknennen wir *Normaldreieck* und bezeichnen seine Seiten durch x_1, x_2, x_3 und seine Ecken durch u_1, u_2, u_3. Den einzelnen Arten C von f entsprechen verschiedene Normaldreiecke. Die der C_1 sind die Wendedreiecke.

Die Normaldreiecke der 5 ersten C sind in endlicher Anzahl vorhanden; die der 5 letzten Arten treten in Schaaren auf. In den folgenden Paragraphen werden die Normaldreiecke für die einzelnen C bestimmt.

§ 11.

Die U_ρ und die V_ρ.

Die Produkte der Seiten und die Produkte der Ecken des Normaldreiecks von C_ρ bezeichnen wir durch

$$U(C_\rho) = x_1x_2x_3\,; \quad V(C_\rho) = u_1u_2u_3\,.$$

Sie haben für die 5 ersten C die Werthe:

$$U(C_1) = \kappa f + \lambda\Delta \quad \text{wo} \quad \kappa^4 - S\kappa^2\lambda^2 - \tfrac{2}{3}T\kappa\lambda^3 - \tfrac{1}{12}S^2\lambda^4 = 0\,;$$
$$U(C_2) = \Delta\,, \quad V(C_2) = s\,;$$
$$V(C_3) = 3St + 5Ts\,;$$
$$V(C_4) = s\,, \quad U(C_4) = s(s) = \tfrac{1}{3}S^2 f + \tfrac{2}{3}T\Delta\,;$$
$$U(C_5) = f\,.$$

Die Cayley'schen Curven der U und V sind die entsprechenden V und U.

Die Berechnung von $U(C_1)$ erfordert die Auflösung einer Gleichung 4ten Grades, die der andern C nur lineare Gleichungen.

§ 12.

Bestimmung der Normaldreiecke der 5 ersten C.

Durch Spaltung der Produkte U und V in ihre Faktoren erhält man die Seiten und die Ecken der Normaldreiecke.

Bei

$$C_1\,,\ C_2\,,\ C_5$$

bedarf man hierzu die Auflösung einer kubischen Gleichung.

Bei C_3 ist

$$V = u_a^3 = 3St + 5Ts = u_1u_2u_3\,,$$

und

$$\Pi = u_1^3 = Ts - St\,,$$

der Cubus der Ecke u_1. Die Ecken u_2 und u_3 findet man durch Spaltung des Produktes:

$$u_1^{-1}u_a^3 = u_2u_3\,,$$

oder auch des Produktes:

$$3u_a^2v_a \cdot v_1^2 - 3u_av_a^2 \cdot u_1v_1 + v_a^2u_1^2 = u_2u_3 \cdot v_1^3\,.$$

Bei C_4 ist

$$U = \tfrac{1}{3}S^2 f + \tfrac{2}{3}T\Delta = p_x^3 = x_1x_2x_3$$

und

$$Tf - S\Delta = x_1^3\,.$$

Die Seiten x_2 und x_2 findet man aus dem Produkte:

$$x_1^{-1}p_x^3 = x_2x_3\,,$$

oder auch aus dem Produkte:

$$3p_x^2p_y \cdot y_1^2 - 3p_xp_y^2 \cdot x_1y_1 + p_y^3x_1^2 = x_2x_3 \cdot y_1^3\,.$$

Im Ganzen erfordert die Aufstellung der Normaldreiecke

1) bei C_1 eine biquadratische und eine kubische Gleichung;
2) bei C_2 und C_5 eine kubische Gleichung;
3) bei C_3 und C_4 eine quadratische Gleichung.

§ 13.

Bestimmung der Normaldreiecke der 5 letzten C.

Zu den 5 letzten C gehören Schaaren von Normaldreiecken:

$$x_1,\ x_2,\ x_3;\quad u_1,\ u_2,\ u_3.$$

Für sie haben die Produkte U und V keine besondere Bedeutung. Wir wollen die bestimmbaren Stücke einzeln berechnen.

1. *Das Normaldreieck von* C_6.

Die Ecke u_1 und die Seite x_2 bestimmt man aus

$$\Theta = -8x_2^2u_1^2,$$

und die Ecke u_3 und die Seite x_3 aus

$$s = -6u_1^2u_3,\quad \Delta = -6x_2^2x_3,$$

oder auch aus

$$3u_xv_x^2v_1 + 2u_1v_x^3 = -6u_3v_1^3,\quad 3\Delta_x\Delta_y^2y_2 - 2\Delta_y^3x_2 = -6x_3y_2^3.$$

2. *Das Normaldreieck von* C_7.

Die Ecke u_3 und die Seite x_1 bestimmt man aus

$$s_3 = -3u_3^3,\quad \Delta = -6x_1^3;$$

x_2 ist eine beliebige Gerade durch u_3; u_1 ist der weitere Schnittpunkt von x_2 mit f; und x_3 ist die Tangente, welche f im Punkte u_1 berührt.

3. *Das Normaldreieck von* C_8.

Die Ecke u_1 und die Seiten x_2 und x_3 bestimmt man aus

$$\Theta = -2u_1^2x_2x_3;$$

x_1 ist eine beliebige Gerade.

4. *Das Normaldreieck von* C_9.

Die Ecke u_1 und die Seite x_2 bestimmt man aus

$$\Theta = -2u_1^2x_2^2,$$

und die Seite x_3 aus

$$f = a_x^3 + 3x_1^2x_3,$$

oder auch aus

$$3a_xa_y^2\cdot y_1 - 2a_y^3\cdot x_1 = 3y_1^3\cdot x_3;$$

x_1 ist eine beliebige Gerade.

5. *Das Normaldreieck von* C_{10}.

Die Seite x_1 bestimmt man aus

$$f = x_1^3.$$

Die Seiten x_2 und x_3 sind beliebige Gerade.

Zur Aufstellung der Normaldreiecke der 5 letzten C ist höchstens eine quadratische Gleichung nöthig.

§ 14.

Die Arten der Hessischen und der Cayley'schen Curven.

In §§ 4 und 5 sind die Hessischen und Cayley'schen Curven der Normalformen aufgestellt worden; wir entnehmen daraus die Arten, denen sie angehören und stellen die Resultate in diesen Tabellen zusammen.

1. *Die Arten der Hessischen Curven.*

Die C_1 und C_2 sind Hessische Curven der C_1.
Die C_3 sind Hessische Curven der C_3.
Die C_4 sind Hessische Curven der C_4.
Die C_5 sind Hessische Curven der C_2 und C_5.
Die C_9 sind Hessische Curven der C_6.
Die C_{10} sind Hessische Curven der C_7.
Die C_6, C_7, C_8 sind keine Hessischen Curven.

2. *Die Arten der Cayley'schen Curven.*

Die C_1 und C_2 sind Cayley'sche Curven der C_1.
Die C_4 sind Cayley'sche Curven der C_3.
Die C_5 sind Cayley'sche Curven der C_2, C_4, und C_5.
Die C_9 sind Cayley'sche Curven der C_6.
Die C_{10} sind Cayley'sche Curven der C_7.
Die C_3, C_6, C_7, C_8 sind keine Cayley'schen Curven.

In den letzten Formeln entstehen die C linker Hand aus denen des § 3 indem man die x durch u ersetzt.

§ 15.

Die Arten der Curven $\bar{\Delta}$ und $\bar{s}$.

Diese Tabellen setzen uns in den Stand die Arten der Curven $\bar{\Delta}$ und $\bar{s}$ anzugeben, welche zu den Curven 3ter Ordnung gehören.

1. *Die Arten der $\bar{\Delta}$.*

Die $\bar{\Delta}$ der C_1 und C_2 sind Curven C_1.
Die $\bar{\Delta}$ der C_3 sind Curven C_3.
Die $\bar{\Delta}$ der C_4 sind Curven C_4.
Die $\bar{\Delta}$ der C_5 sind Curven C_2 und C_5.
Die $\bar{\Delta}$ der C_9 sind Curven C_6.
Die $\bar{\Delta}$ der C_{10} sind Curven C_7.
Zu den Curven

$$C_6,\ C_7,\ C_8$$

gehören keine $\bar{\Delta}$.

2. *Die Arten der* $\bar{s}$.

Die $\bar{s}$ der C_1 und C_2 sind Curven C_1.
Die $\bar{s}$ der C_4 sind Curven C_3.
Die $\bar{s}$ der C_5 sind Curven C_2, C_4, C_5.
Die $\bar{s}$ der C_9 sind Curven C_6.
Die $\bar{s}$ der C_{10} sind Curven C_7.
Zu den Curven

$$C_3,\ C_6,\ C_7,\ C_8$$

gehören keine $\bar{s}$.

§ 16.

Definition der Moduln der Curven $\bar{\Delta}$ *und* $\bar{s}$.

Die linken Seiten der Gleichungen der Curven:

$$\bar{\Delta}(f) = 0\,;\quad \bar{s}(f) = 0\,,$$

sind bis auf Faktoren bestimmt, die im Allgemeinen von den Coefficienten von f abhängen. Ist

$$\bar{\Delta}(f) = v\,;\quad \bar{s}(f) = w\,,$$

so hat man

$$\Delta(v) = C_1 f\,;\quad s(w) = C_2 f.$$

Die Faktoren C_1 und C_2 nennen wir die *Moduln* von $\bar{\Delta}$ und $\bar{s}$ und bezeichnen sie durch

$$C_1 = M\big(\bar{\Delta}(f)\big)\,;\quad C_2 = M\big(\bar{s}(f)\big)\,.$$

§ 17.

Die Werthe der $\bar{\Delta}$ *der Normalformen.*

$$\begin{aligned}
\bar{\Delta}(C_1) &= x_1^3 + x_2^3 + x_3^3 + 6\mu x_1x_2x_3 && \left(m = \tfrac{1+2\mu^3}{-6\mu^2}\right);\\
\bar{\Delta}(C_2) &= x_1^3 + x_2^3 + x_3^3 + 6\mu x_1x_2x_3 && (0 = 1 + 2\mu^3);\\
\bar{\Delta}(C_3) &= x_2^3 + x_3^3 - 18x_1x_2x_3\,;\\
\bar{\Delta}(C_4) &= x_1^3 - 18x_1x_2x_3\,;\\
\bar{\Delta}(C_5) &= -18x_1x_2x_3 \text{ und } l_1x_1^3 + l_2x_2^3 + l_3x_3^3\,;\\
\bar{\Delta}(C_9) &= 3x_1x_2^2 + \lambda x_3^3\,;\\
\bar{\Delta}(C_{10}) &= 3x_1x_2^2 + 3\lambda\, x_1^2x_3\,.
\end{aligned}$$

Es entsprechen

den Curven C_1 3 Curven $\bar{\Delta}$,
den Curven C_2 3 Curven $\bar{\Delta}$,
den Curven C_3 1 Curve $\bar{\Delta}$,
den Curven C_4 1 Curve $\bar{\Delta}$,
den Curven C_5 1 Curve $\bar{\Delta}$,

und ausserdem

den Curven C_5, C_9, C_{10} Schaaren von Curven $\bar{\Delta}$.

Die Bestimmung der $\bar{\Delta}$ der Normalformen erfordert bei den C_1 und C_2 die Auflösung einer kubischen Gleichung; bei C_2 hat sie numerische Coefficienten. Die übrigen $\bar{\Delta}(C)$ sind linear bestimmbar.

§ 18.

Die Werthe der $\bar{s}$ der Normalformen.

$$\bar{s}(C_1) = x_1^3 + x_2^3 + x_3^3 + 6\mu x_1x_2x_3 \qquad \left(m = \frac{4\mu^3 - 1}{-6\mu}\right);$$

$$\bar{s}(C_2) = x_1^3 + x_2^3 + x_3^3 + 6\mu x_1x_2x_3 \qquad (4\mu^3 = 1);$$

$$\bar{s}(C_4) = x_2^3 - \tfrac{2}{3}\mu^2 x_3^3 + 6\mu x_1x_2x_3;$$

$$\bar{s}(C_5) = l_1x_1^3 + l_2x_2^3 + l_3x_3^3 \text{ und } \lambda x_1^3 + 6\mu x_1x_2x_3 \qquad (4\mu^3 = 1);$$

$$\bar{s}(C_9) = 3x_1^2x_2 + \lambda x_3^3;$$

$$\bar{s}(C_{10}) = 3x_2^2x_3 + 3\lambda x_1x_3^2.$$

In diesen Formeln entstehen die C aus denen des § 3, indem man die x durch die u ersetzt.

Den Curven C_1 und C_2 entsprechen je 3 Curven $\bar{s}$, den übrigen C Curvenschaaren.

Die Bestimmung der

$$\bar{s}(C_1),\ \bar{s}(C_2),\ \bar{s}(C_3)$$

erfordert die Auflösung kubischer Gleichungen, von denen die beiden letzten numerische Coefficienten haben. Die übrigen $\bar{s}(C)$ sind linear bestimmbar.

§ 19.

Die Werthe der Moduln der $\bar{\Delta}(C)$ und $\bar{s}(C)$.

1. *Die Moduln der $\bar{\Delta}(C)$.*

$$M(\bar{\Delta}(C_1)) = -6\mu^2;$$

$$M(\bar{\Delta}(C_2)) = -6\mu^2;$$

$$M(\bar{\Delta}(C_3)) = -54;$$

$$M(\bar{\Delta}(C_4)) = -54;$$

$$M(\bar{\Delta}(C_5)) = -54 \text{ und } l_1l_2l_3;$$

$$M(\bar{\Delta}(C_9)) = -2\lambda;$$

$$M(\bar{\Delta}(C_{10})) = -6\lambda^2.$$

2. *Die Moduln der* $\bar{s}(C)$.

$$M\left(\bar{s}(C_1)\right) = -6\mu;$$
$$M\left(\bar{s}(C_2)\right) = -6\mu;$$
$$M\left(\bar{s}(C_4)\right) = 4\mu^3;$$
$$M\left(\bar{s}(C_5)\right) = -l_1l_2l_3 \text{ und } 1;$$
$$M\left(s(C_9)\right) = -2\lambda;$$
$$M\left(\bar{s}(C_{10})\right) = -6\lambda.$$

§ 20.

Bestimmung der $\bar{\Delta}$ *und* $\bar{s}$ *mit Hülfe der Normalformen.*

Um die $\bar{\Delta}$ und $\bar{s}$ einer Curve 3ter Ordnung f zu erhalten kann man sich der Normalformen bedienen. Man transformirt f in die Normalform und berechnet von dieser die gesuchten Curven. Hierbei sind diese Gleichungen erforderlich:

1. Für die C_1 1 biquadratische und 2 kubische Gleichungen.
2. Für die C_2 2 kubische Gleichungen.
3. Für die C_3 1 quadratische Gleichung.
4. Für die C_4 1 quadratische Gleichung.
5. Für die C_5
   1) bei Bestimmung der $\bar{\Delta}$ eine kubische Gleichung;
   2) bei Bestimmung der $\bar{s}$ 2 kubische Gleichungen, deren eine numerische Coefficienten hat.
6. Für die C_9 und C_{10} bedarf man keine Gleichungen höheren Grades.

Für die Arten

$$C_3,\ C_4,\ \cdots,\ C_{10}$$

empfiehlt sich die Benutzung der Normalformen.

§ 21.

Bestimmung der $\bar{\Delta}$ *und* $\bar{s}$ *ohne Hülfe der Normalformen.*

Bei den C_1 und C_2 ist der Durchgang durch die Normalformen nicht zweckmässig, weil er die Auflösung überflüssiger Gleichungen verlangt.

Die C_1 und C_2 haben keinen Doppelpunkt, bei ihnen ist

$$R = T^2 - \tfrac{1}{6}S^3 \neq 0.$$

Bei ihnen wollen wir die $\bar{\Delta}$ und $\bar{s}$ direkt berechnen, indem wir der Reihe nach diese Formen bestimmen:

1. Die absoluten Invarianten:

$$u(\bar{\Delta}),\ u(\bar{s}) \qquad (u = S^3/T^2).$$

2. Die Invarianten:

$$T(\bar{\Delta}),\ S^3(\bar{\Delta}),\ R(\bar{\Delta}),\ T(\bar{s}),\ S^3(\bar{s}),\ R(\bar{s}).$$

3. Die $\bar{\Delta}$ und $\bar{s}$ selbst.

Nur die Berechnung der absoluten Invarianten erfordert die Auflösung einer kubischen Gleichung, die übrigen Formen werden daraus linear bestimmt.

§ 22.

Einige Covarianten und Invarianten von Δ und s.

Die einfachsten Covarianten und Invarianten von f sind

$$\Delta,\ s,\ t,\ S,\ T,\ R.$$

Wir entnehmen die Werthe einiger von ihnen, für Δ und s gebildet, dem Aufsatze, Mathematische Annalen, Bd. 6.

$$(1)\qquad \begin{cases} 12\,\Delta(\Delta) = S^2 f - 4T\Delta; \\ 12\,S(\Delta) = 8T^2 - S^3; \\ 72\,T(\Delta) = 3S^3T - 16T^3; \\ 12^3 R(\Delta) = -S^6 R; \\ 3\ s(s) = S^2 f + 2T\Delta; \\ 9\ t(s) = 2S^2 Tf + (S^3 - 2T^2)\Delta; \\ 3\,S(s) = S^3 + 2T^2; \\ 9\,T(s) = 3S^3T - 2T^3. \end{cases}$$

$$(2)\qquad R(s) = -\tfrac{2}{9} S^3 R^2.$$

Wir setzen daraus diese Formeln zusammen.

$$(3)\qquad u(\Delta) = -3\,\frac{(u-8)^3}{(3u-16)^2};$$

$$(4)\qquad T = -6\,\frac{T(\Delta)}{S(\Delta)}\,\frac{u-8}{3u-16};$$

$$(5)\qquad S^2 f = 12\Delta(\Delta) + 4T\Delta;$$

$$(6)\qquad u(s) = 3\,\frac{(u+2)^3}{(3u-2)^2};$$

$$(7)\qquad T = 3\,\frac{T(s)}{S(s)}\,\frac{u+2}{3u-2};$$

$$(8)\qquad S^2 Rf = (T^2 - \tfrac{1}{2}S^3)\,s(s) + 3T\,t(s).$$

§ 23.

Die den C_1 und C_2 entsprechenden Curven $\bar{\Delta}$.

Ersetzt man in den Formeln 1, 3, 4, 5 des §22 die Form Δ durch f, so gehen

$$S,\quad T,\quad R,\quad u,\quad f,\quad \Delta,\quad S(\Delta),\quad T(\Delta),\quad R(\Delta),\quad u(\Delta),\quad \Delta(\Delta)$$

in

$$S(\bar{\Delta}),\quad T(\bar{\Delta}),\quad R(\bar{\Delta}),\quad u(\bar{\Delta}),\quad \bar{\Delta},\quad f,\quad S,\quad T,\quad R,\quad u,\quad \Delta$$

über und man hat

$$12^3 R = -\, S^6(\bar{\Delta}) R(\bar{\Delta}) \neq 0\,,$$

$$u = -\,3\,\frac{\big(u(\bar{\Delta})-8\big)^3}{\big(3u(\Delta)-16\big)^2},$$

$$T(\bar{\Delta}) = -\,6\,\frac{T}{S}\,\frac{u(\bar{\Delta})-8}{3u(\Delta)-16},$$

$$S^2(\bar{\Delta})\bar{\Delta} = 12\Delta + 4\,T(\bar{\Delta}) f. \tag{9}$$

§ 24.

Die den C_1 und C_2 entsprechenden Curven $\bar{s}$.

Ersetzt man in den Formeln 2, 6, 7, 8 des § 22 die Form s durch f, so gehen

$$S,\quad T,\quad R,\quad u,\quad f,\quad s,\quad S(s),\quad T(s),\quad R(s),\quad u(s),\quad s(s),\quad t(s)$$

in

$$S(\bar{s}),\quad T(\bar{s}),\quad R(\bar{s}),\quad u(\bar{s}),\quad \bar{s},\quad f,\quad S,\quad T,\quad R,\quad u,\quad s,\quad t$$

über und man hat

$$R = -\,\tfrac{2}{9} S^3(\bar{s}) R^2(\bar{s}) \neq 0\,,$$

$$u = 3\,\frac{\big(u(\bar{s})+2\big)}{\big(3u(\bar{s})-2\big)^2},$$

$$T(\bar{s}) = 3\,\frac{T}{S}\,\frac{u(\bar{s})+2}{3u(\bar{s})-2}\,,$$

$$S^3(\bar{s}) R(\bar{s}) \bar{s} = \big(T^2(\bar{s}) - \tfrac{1}{2} S(\bar{s})\big) s + 3\,T(\bar{s}) t\,. \tag{10}$$

§ 25.

Endgültige Umgränzung der Sätze über Tripel von Curven.

Den Sätzen, auf die sich die Untersuchungen des Herrn WHITE beziehen, nämlich: "Es giebt Tripel von Curven welche mit f die Hessischen und Tripel welche mit f die Cayley'schen gemein haben," entspricht nach (9) und (10) (§§ 23, 24) der Satz:

Zu jeder Curve C_1 und C_2 gehören 3 Curven $\bar{\Delta}$ und 3 Curven $\bar{s}$.

APPLICATION OF A METHOD OF D'ALEMBERT TO THE PROOF OF STURM'S THEOREMS OF COMPARISON*

BY

MAXIME BÔCHER

Of the many theorems contained in STURM's famous memoir in the first volume of Liouville's Journal (1836), p. 106, two, which I have called the *Theorems of Comparison*, may be regarded as most fundamental. I have recently shown† how the methods which STURM used for establishing these theorems can be thrown into rigorous form.

In the present paper I propose to prove these theorems by a simpler‡ and more direct method. This method was suggested to me by a passage, to which Professor H. BURKHARDT kindly called my attention, in one of D'ALEMBERT's papers on the vibration of strings.§ D'ALEMBERT's fundamental idea, and indeed all that I here preserve of his method, consists in replacing the linear differential equations by RICCATI's equations.‖

§ 1.

We begin then by considering two of RICCATI's equations:

$$(R_1) \qquad \frac{d\omega}{dx} = A_1(x) + C_1(x)\,\omega^2,$$

$$(R_2) \qquad \frac{d\omega}{dx} = A_2(x) + C_2(x)\,\omega^2,$$

where A_1, A_2, C_1, C_2 are, throughout the interval $a \leqq x \leqq b$, continuous functions¶ of x satisfying the inequalities:

$$A_2(x) \geqq A_1(x), \qquad C_2(x) \geqq C_1(x).$$

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* Presented to the Society June 29, 1900. Received for publication July 8, 1900.

† Bulletin of the American Mathematical Society, April, 1898, p. 295, and December, 1899, p. 100.

‡ Simpler, at least, if we wish to establish the theorems in all their generality.

§ Memoirs of the Berlin Academy, vol. 70 (1763), p. 242.

‖ Sturm also in the paper quoted (p. 159) uses Riccati's equations, but only incidentally, and for quite a different purpose.

¶ All the quantities used in this paper are real.

We assume that (R_1) and (R_2) have solutions, $\omega_1(x)$ and $\omega_2(x)$ respectively, which are continuous throughout the interval $a \leqq x \leqq b$.

LEMMA I. *If* $A_2(c) > A_1(c)$ *and* $\omega_2(c) = \omega_1(c)$ $(a \leqq c < b)$, *then a positive* ϵ *exists such that*

$$\omega_2(x) > \omega_1(x) \qquad (c < x < c + \epsilon).$$

For by comparing (R_1) and (R_2) we see that:

$$\omega_2'(c) > \omega_1'(c).^*$$

Accordingly if $f(x) = \omega_2(x) - \omega_1(x)$, we have $f(c) = 0$, $f'(c) > 0$, from which it follows that, for values of x a little greater than c, $f(x) > 0$.

THEOREM I. *If* $\omega_2(a) \geqq \omega_1(a)$, *then* $\omega_2(x) \geqq \omega_1(x)$ $(a < x \leqq b)$.

This theorem we prove first on the supposition that A_1 and A_2 are not equal at any point of the interval $a \leqq x < b$.

First we notice that the theorem is surely true in a small neighborhood of the point a $(a < x < a + \epsilon)$. This we see at once when $\omega_2(a) > \omega_1(a)$ from the continuity of ω_1 and ω_2, and when $\omega_2(a) = \omega_1(a)$ from lemma I.

Let us now consider the interval $a < x < c$ where c has the largest value $(c \leqq b)$ such that our theorem holds throughout this whole interval. If $c < b$ it is obvious that $\omega_2(c) = \omega_1(c)$, and, therefore, by lemma I $\omega_2(x) > \omega_1(x)$ $(c < x < c + \epsilon)$. This is contrary to hypothesis, as it shows that c might have had a larger value. Therefore $c = b$ and our theorem, in the special case we are considering, is proved.

In order to establish the theorem in the general case we first prove

LEMMA II. *If* $\omega_2(c) = \omega_1(c)$ $(a \leqq c < b)$, *then a positive* ϵ *exists such that:*

$$\omega_2(x) \geqq \omega_1(x) \qquad (c < x < c + \epsilon).$$

To prove this we consider the new RICCATI'S equation:

$$(\bar{R}_2) \qquad \frac{d\omega}{dx} = A_2(x) + \lambda + C_2(x)\,\omega^2,$$

where λ, which we shall regard as a parameter, is independent of x. Let $\bar{\omega}_2(x, \lambda)$ be the solution of this equation which has at the point $x = c$ the same value which $\omega_2(x)$ has there. If now we restrict λ to the interval $0 \leqq \lambda \leqq k$ (where k is any positive constant), there exists a positive ϵ independent of λ such that $\bar{\omega}_2(x, \lambda)$ is a continuous function of (x, λ) when $c \leqq x < c + \epsilon$.† In particular we have for every value of x in this interval:

$$\lim_{\lambda=0} \bar{\omega}_2(x, \lambda) = \omega_2(x).$$

---

* We use accents here and in what follows to denote differentiation.

† This follows at once from the analysis on p. 303 of PICARD'S *Traité d'Analyse*, vol. 2.

Let us now compare $\bar{\omega}_2(x, \lambda)\,(\lambda > 0)$ with $\omega_1(x)$ by means of that part of theorem I which we have already proved. This gives

$$\bar{\omega}_2(x, \lambda) > \omega_1(x) \qquad (c < x < c + \epsilon).$$

By taking the limit for $\lambda = 0$ the truth of our lemma follows at once.

We can now use lemma II to prove theorem I in the general case as we used lemma I to prove it in the special case. As the reasoning is precisely the same, we will not repeat it.

The following modification of theorem I will be of use to us:

THEOREM I'. *If* $\omega_1(b) \geqq \omega_2(b)$ *then* $\omega_1(x) \geqq \omega_2(x)$ $(a \leqq x < b)$.

This theorem may most readily be deduced from theorem I by using the transformation $\bar{x} = a + b - x$ which has the effect of interchanging the points a and b and of changing the sign of ω'.

We will now make theorem I a little more precise as follows:

THEOREM II. *If* $\omega_2(a) \geqq \omega_1(a)$ *then* $\omega_2(x) > \omega_1(x)$ $(a < x \leqq b)$ *provided that when* $\omega_2(a) = \omega_1(a)$ *one excludes the two cases:*

1) *throughout a certain neighborhood of a* $A_2 = A_1$, $C_2 = C_1$.

2) *in every neighborhood of a a point p exists such that throughout a certain neighborhood of p* $\omega_2 = \omega_1 = 0$.*

We have here merely to prove that if $\omega_2 = \omega_1$ at any point of the interval $a < x < b$, one or the other of the cases here excluded must occur. Let c be a point at which $\omega_2 = \omega_1$. Then theorem I' tells us that

$$\omega_2(x) \leqq \omega_1(x) \qquad (a < x < c),$$

while by theorem I:

$$\omega_2(x) \geqq \omega_1(x) \qquad (a < x < c).$$

Accordingly:

$$\omega_2(x) = \omega_1(x) \qquad (a < x < c);$$

and since ω_1 and ω_2 are continuous

$$\omega_2(a) = \omega_1(a).$$

Therefore:

$$\omega_2'(x) = \omega_1'(x) \qquad (a \leqq x < c).$$

By subtracting (R_1) from (R_2) we now get:

$$(A) \qquad 0 = A_2(x) - A_1(x) + \big(C_2(x) - C_1(x)\big)\omega_1^2 \qquad (a \leqq x < c).$$

Since $A_2 - A_1$, $C_2 - C_1$, and ω_1^2 can none of them be negative

$$A_2(x) = A_1(x) \qquad (a \leqq x < c).$$

* This second case can obviously occur only when $A_1 = A_2 = 0$ throughout the neighborhood of p.

Moreover *either* throughout a certain neighborhood of a $C_2 = C_1$ (in which case we have exception 1) *or* there are points in every neighborhood of a at which $C_2 > C_1$. Throughout a certain neighborhood of any one of these points we must (on account of the continuity of C_1 and C_2) also have $C_2 > C_1$, and therefore on account of the equality (A) $\omega_1 = \omega_2 = 0$ and we have the exceptional case 2.

§ 2.

Instead of assuming, as we did in the last section, that ω_1 and ω_2 are continuous throughout the interval $a \leqq x \leqq b$, we will now merely assume that they are continuous throughout the interval $a < x \leqq b$, and that they are either continuous at the point a or become positively or negatively infinite there.

THEOREM III. *If* $\omega_2(a) = +\infty$, *then* $\omega_2(x) > \omega_1(x)$ $(a < x \leqq b)$, *provided that, if* $\omega_1(a) = +\infty$, *we exclude the case in which throughout a certain neighborhood of* a $A_2 = A_1$, $C_2 = C_1$.

If we can prove this theorem for the immediate neighborhood of a, its truth for the whole interval $a < x \leqq b$ will follow at once by an application of theorem II to the remainder of this interval. Moreover except in the case $\omega_1(a) = +\infty$ the theorem follows for the neighborhood of a from the mere continuity of ω_1 and ω_2.

It remains therefore only to prove the theorem for the neighborhood of a when $\omega_1(a) = +\infty$. For this purpose we take the neighborhood of a so short that neither ω_1 nor ω_2 vanishes in it, and we introduce into the equations (R_1) and (R_2) the new dependent variables:

$$\bar{\omega}_1 = -\frac{1}{\omega_1}, \qquad \bar{\omega}_2 = -\frac{1}{\omega_2},$$

getting as the equations satisfied by $\bar{\omega}_1$ and $\bar{\omega}_2$:

$$\frac{d\bar{\omega}}{dx} = C_1(x) + A_1(x) \cdot \bar{\omega}^2,$$

$$\frac{d\bar{\omega}}{dx} = C_2(x) + A_2(x) \cdot \bar{\omega}^2.$$

These equations having the same form as (R_1) and (R_2), and $\bar{\omega}_1$, $\bar{\omega}_2$ being solutions of them which are continuous in the neighborhood of a, we can at once apply theorem II to them. Since $\bar{\omega}_1$ and $\bar{\omega}_2$ do not vanish in the neighborhood of a, exception 2 cannot occur. We see thus that, except when throughout a certain neighborhood of a $A_1 = A_2$ and $C_1 = C_2$,

$$\bar{\omega}_2(x) > \bar{\omega}_1(x) \qquad (a < x < a + \varepsilon).$$

Therefore, since ω_1 and ω_2 are both positive:

$$\omega_2(x) > \omega_1(x) \qquad (a < x < a + \varepsilon).$$

Finally, if we drop the requirement that ω_1 and ω_2 be continuous at b, theorems II and III will obviously still hold throughout the interval $a < x < b$. We add also the following theorem:

THEOREM IV. *If ω_1 and ω_2 are continuous at a and $\omega_2(a) \geqq \omega_1(a)$, or if $\omega_2(a) = +\infty$ while ω_1 is either continuous at a or becomes positively or negatively infinite there, then, provided we exclude the exceptional cases of theorems II and III, we cannot have $\omega_2(b) = -\infty$.*

For we should also have $\omega_1(b) = -\infty$ on account of the inequality

$$\omega_2(x) > \omega_1(x) \qquad (a < x < b).$$

If then we let as above:

$$\bar{\omega}_1 = -\frac{1}{\omega_1}, \qquad \bar{\omega}_2 = -\frac{1}{\omega_2},$$

in the neighborhood of b these functions are continuous and vanish only at b. Since in the neighborhood of b they satisfy the inequality $\bar{\omega}_2 > \bar{\omega}_1$, they should also by theorem II satisfy this inequality at b; whereas they are equal there. We are thus led to a contradiction and our theorem is proved.

§ 3.

We now turn to the two linear equations:

$$(L_1) \qquad \frac{d\left(K_1(x)\frac{dy}{dx}\right)}{dx} + G_1(x)\, y = 0,$$

$$(L_2) \qquad \frac{d\left(K_2(x)\frac{dy}{dx}\right)}{dx} + G_2(x)\, y = 0.$$

Here we assume that K_1, K_2, G_1, G_2 are throughout the interval $a \leqq x \leqq b$ continuous functions of x which satisfy the inequalities:

$$K_2(x) \geqq K_1(x) > 0, \qquad G_2(x) \geqq G_1(x).$$

Moreover we assume that K_1 and K_2 have continuous first derivatives throughout this interval.

Let $y_1(x)$ and $y_2(x)$ be solutions of (L_1) and (L_2) respectively. These functions are, of course, continuous together with their first derivatives throughout the interval $a \leqq x \leqq b$. We shall further assume that if $y_2(a) \neq 0$ then:

$$y_1(a) \neq 0 \quad \text{and} \quad K_2(a)\frac{y_2'(a)}{y_2(a)} \geqq K_1(a)\frac{y_1'(a)}{y_1(a)}.$$

Here again there are two special cases which must be excluded:

1) If $y_2(a)=y_1(a)=0$, or if $y_2(a)\neq 0$ and $K_2(a)y_2'(a)/y_2(a)=K_1(a)y_1'(a)/y_1(a)$, we exclude the case in which throughout a certain neighborhood of a $K_1=K_2$ and $G_1=G_2$.

2) If $y_2'(a)=y_1'(a)=0$, we exclude the case in which in every neighborhood of a a point p exists such that throughout a certain neighborhood of p $y_1'=y_2'=0$. This exception includes the case in which y_1 and y_2 vanish at all points of the interval $a\leqq x\leqq b$. Apart from this trivial case we note that the case we here exclude can occur only when G_1 and G_2 both vanish throughout the neighborhoods of the points p in question.*

These restrictions having been made we let:

$$\omega_1=K_1\frac{y_1'}{y_1},\qquad \omega_2=K_2\frac{y_2'}{y_2}.$$

The functions ω_1 and ω_2 then satisfy the equations (R_1) and (R_2) respectively if

$$A_1=-G_1,\quad A_2=-G_2,\quad C_1=-1/K_1,\quad C_2=-1/K_2.$$

We therefore get at once from theorems II and III:

THEOREM V. *If neither y_1 nor y_2 vanishes in the interval $a<x<b$ then:*

$$K_2\frac{y_2'}{y_2}>K_1\frac{y_1'}{y_1}\qquad (a<x<b).$$

From theorem IV we get:

THEOREM VI. *If y_2 does not vanish in the interval $a<x<b$ and if $y_2(b)=0$, then y_1 has at least one root in the interval $a<x<b$.*

§ 4.

The two theorems of the last section are nothing but special cases of Sturm's theorems of comparison, and from them the general theorems will now be deduced.

STURM'S FIRST THEOREM OF COMPARISON. *If y_2 has n roots in the interval $a<x\leqq b$, then y_1 has at least n roots there, and the kth root of y_1 measured from a is less than the kth root of y_2.*

Let $x_1, x_2, \cdots, x_n$ be the roots of y_2 $(a<x_1<x_2<\cdots<x_n\leqq b)$. The truth of the theorem follows at once when we notice that by theorem VI in the interval $a<x<x_1$ and also in each of the intervals

$$x_i\leqq x<x_{i+1}\qquad (i=1, 2, \cdots, n-1)$$

there lies at least one root of y_1.

* This second exception is not ordinarily mentioned. If G_1 and G_2 are to be analytic throughout the interval $a<x<b$, this case can arise only if G_1 and G_2 vanish identically, in which case $y=k$ is a solution of both equations, no matter what values K_1 and K_2 have.

STURM'S SECOND THEOREM OF COMPARISON. *If c is a point of the interval $a < x < b$ where neither y_1 nor y_2 is zero, and if in the interval $a < x < c$, y_1 and y_2 have the same number n of roots, then:*

$$K_2(c)\frac{y_2'(c)}{y_2(c)} > K_1(c)\frac{y_1'(c)}{y_1(c)}.$$

If $n = 0$ this reduces to theorem V. If $n > 0$ let x_n be the nth root of y_2 measured from a. Then since, by STURM'S first theorem of comparison, y_1 has at least n roots in the interval $a < x < x_n$, and by hypothesis it has just n roots in the interval $a < x < c$, y_1 can have no root in the interval $x_n \leqq x \leqq c$. We may therefore apply theorem V to the interval $x_n < x < c + \epsilon$ and STURM'S theorem follows at once.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

TWO PLANE MOVEMENTS GENERATING QUARTIC SCROLLS*

BY

E. M. BLAKE

The object of this paper is to give the results of a study of the point and line loci of two plane movements. The former have received some consideration as the references given below will show, but the latter it is believed have not been described before.

I. The Degenerate Case of Three-bar Movement whose Centrodes are Limaçons.

§1. *Definition of the movement. Point loci.*

In space regarded as fixed let a system of rectangular coördinates x, y, z be established, whose origin is O. Denote the xy-plane by σ. Upon it moves a plane σ_1 carrying two other planes, the three being mutually perpendicular. By means of them we establish a system of coördinates x_1, y_1, z_1 whose origin will be denoted by O_1. The axes of x_1 and y_1 are taken in σ_1. Denote the point $(a, 0, 0)$ of fixed space by A, and $(a_1, 0, 0)$ of the moving space by A_1, where a and a_1 are positive. The movement to be studied is defined by the conditions that A_1 is to remain at the distance a_1 from O, and O_1 at the distance a from A. This definition implies that the limaçon

$$\left(x_1^2 + y_1^2 - \frac{2a^2a_1}{a^2 - a_1^2}x_1\right)^2 = \left(\frac{2aa_1^2}{a^2 - a_1^2}\right)^2 (x_1^2 + y_1^2),$$

in σ_1, rolls upon the related one:

$$\left(x^2 + y^2 - \frac{2aa_1^2}{a_1^2 - a^2}x\right)^2 = \left(\frac{2a^2a_1}{a_1^2 - a^2}\right)^2 (x^2 + y^2),$$

in σ.† These centrodes will be denoted by c_1 and c respectively. The real double points of the centrodes are O_1 and O, and A_1 and A are foci.

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* Extract from a paper presented to the Society December 28, 1899, under the title: *Upon plane movements whose point loci are of order not greater than four.* Received for publication April 26, 1900.

† S. Roberts, Proceedings London Mathematical Society, vol. 3 (1871), p. 95. Burmester, *Lehrbuch der Kinematik*, vol. 1, pp. 306-308.

The loci of the points of σ_1 are bicircular unicursal quartics and pedals of conics. The equation of the locus of the point (x_1, y_1) is

$$\{a_1(x^2+y^2)+a(xx_1+yy_1-2a_1x)\}^2+(a^2-a_1^2)(x_1y-xy_1)^2 \\ -\{a(x_1^2+y_1^2)+a_1(xx_1+yy_1-2ax_1)\}^2=0;$$

the point

$$x=-\frac{ax_1(x_1^2+y_1^2-2a_1x_1)}{a_1(x_1^2+y_1^2)};\quad y=-\frac{ay_1(x_1^2+y_1^2-2a_1x_1)}{a_1(x_1^2+y_1^2)},$$

is its real double point.*

The forms of the point loci are given by the following theorem, which is not difficult to demonstrate. When $a<a_1$ (in which case c_1 has a real conjugate point) all points within c_1 describe curves consisting of two loops one within the other and joined at a node. The locus of any point without c_1 is a closed loop with conjugate point within, and that of any point upon c_1 a loop with cusp pointing inward. All nodes fall upon σ between the two loops of c, all cusps upon c, and all conjugate points either within the smaller loop of c or without the curve according as the tracing point is within or without the circle $x_1^2+y_1^2-2a_1x_1=0$. Any point upon this circle describes a curve whose double point falls at O. The circle $x^2+y^2-2ax=0$ is twice completely described by O_1 and the locus of A_1 is $x^2+y^2=a_1^2$ once completely described and $(x-a)^2+y^2=0$. When $a>a_1$ (in which case c_1 has a real node) all points between the two loops of c_1 describe curves having two loops mutually exterior joined at a node. The locus of any point within the smaller loop of c_1 or without the curve consists of a closed loop with a conjugate point without, and that of any point upon c_1 a loop with a cusp pointing outward. All nodes fall upon σ within c, all cusps upon c, and all conjugate points without c and either within or without the circle $x^2+y^2-2ax=0$ according as the tracing point is within the smaller loop of c_1 or without c_1. Any point upon the circle $x_1^2+y_1^2-2a_1x_1=0$ describes a curve whose double point falls at O. An arc (only) of the circle $x^2+y^2-2ax=0$ is twice described by O_1 and the locus of A_1 is $x^2+y^2=a_1^2$ once completely described and $(x-a)^2+y^2=0$.

§ 2. *Line loci.*

The equations and forms of the loci of the straight lines carried by σ_1 and oblique to it, can be readily determined by the method already used by the author.† As the equations can be immediately written down we omit them and

*S. ROBERTS, Proceedings London Mathematical Society, vol. 2 (1869), pp. 133–135 and vol. 3 (1871), pp. 88–99.

CAYLEY, ibid., vol. 3 (1871), pp. 100–106.

†American Journal of Mathematics, vol. 21 (1899), pp. 257–269.

give the following theorems on the forms of the scrolls. The projection of the generatrix upon σ_1 is denoted by l' and the distance of l' from O_1 by p.

When $p \neq 0$ the line-loci are unicursal quartic scrolls having a nodal cubical ellipse. They are of the sixteen types shown in the following table.

Type.	Value of $a : a_1$.	Position of l'.	Holgate's* F_4^1 subform
I	$a < a_1$	l' intersects c_1 in four imaginary points.	17
II	$a < a_1$	l' intersects c_1 in two real distinct and two imaginary points.	11
III	$a < a_1$	l' intersects c_1 in two coincident and two imaginary points.	13
IV	$a < a_1 < 2a$	l' is tangent to c_1 where it is convex inward.	4
V	$a < a_1 < 2a$	l' intersects c_1 in four real distinct points.	1
VI	$a < a_1 < 2a$	l' is the double tangent of c_1.	3
VII	$a < a_1 < 2a$	l' is tangent to c_1 where it is concave inward and intersects it in two real distinct points.	2
VIII	$a < a_1 < 2a$	l' is tangent to c_1 at an inflexion.	8
IX	$a_1 = 2a$	l' has a contact of 3d order with c_1.	10
X	$a > a_1$	l' intersects c_1 in four imaginary points.	17
XI	$a > a_1$	l' intersects c_1 in two real distinct and two imaginary points.	11
XII	$a > a_1$	l' intersects c_1 in two coincident and two imaginary points.	13
XIII	$a > a_1$	l' is tangent to the inner loop of c_1.	4
XIV	$a > a_1$	l' intersects c_1 in four real distinct points.	1
XV	$a > a_1$	l' is the double tangent of c_1.	3
XVI	$a > a_1$	l' is tangent to the outer loop of c_1 and intersects it in two other real distinct points.	2

When $p = 0$ the line-loci are unicursal quartic scrolls having a nodal circle and an intersecting nodal straight line. They are of four types as follows:

Type.	Value of $a : a_1$	Position of l'.	Holgate's F_4^1 subform.
XVII	$a < a_1$		3
XVIII	$a > a_1$	l' intersects both loops of c_1.	1
XIX	$a > a_1$	l' intersects the outer loop of c_1 in two points.	1
XX	$a > a_1$	l' is tangent to c_1 at its node.	4

II. Conchoidal Movement.

§ 3. *Definition of the movement. Point loci.*

To define the movement, we establish as before a system of rectangular coördinates x, y, z in fixed space, another system x_1, y_1, z_1 in the moving space, and then condition σ_1 to move upon σ so that the axis of x_1 always passes through

*This refers to the "species" and "subforms" of the classification of unicursal quartic scrolls as given in the two articles: Holgate, *On certain ruled surfaces of the fourth order*, American Journal of Mathematics, vol. 15 (1893), pp. 344–386; *Note* etc., ibid., vol 22 (1900), pp. 27–30; where a general description of the above surfaces can be found.

the fixed point $(0, a, 0)$ and the point $(0, a_1, 0)$ remains upon the axis of x. The loci of the points of σ_1 are unicursal quartics having one double point at infinity on the axis of x. The equation of the locus of (x_1, y_1) is

$$y^2[x^2 + (y-a)^2] - [x(y_1 - a_1) + x_1(y-a)]^2 + 2x_1y_1xy - 2y_1y(y_1 - a_1)(y-a) + y_1^2[x_1^2 + (y_1 - a_1)^2] = 0.*$$

Besides the one mentioned the curve has the two double points:

$$x = \frac{x_1}{2(y_1 - a_1)}\left[a \mp \sqrt{a^2 + 4y_1(y_1 - a_1)}\right],$$

$$y = \tfrac{1}{2}\left[a \pm \sqrt{a^2 + 4y_1(y_1 - a_1)}\right].$$

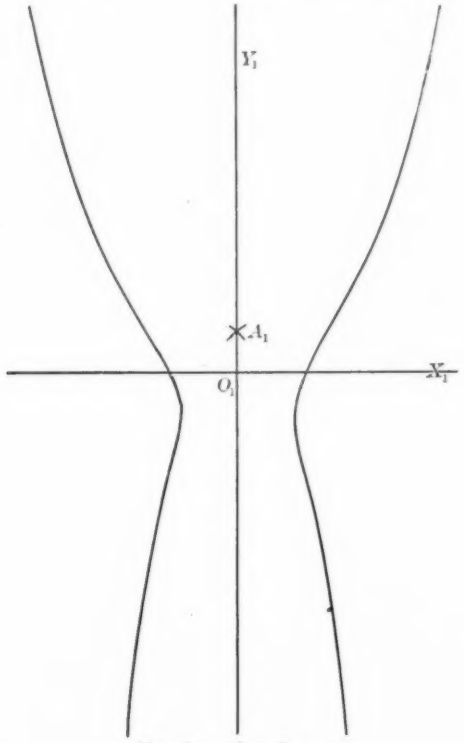

FIG. 1. c_1 for $a > a_1$.

*This equation in another form is due to S. ROBERTS (Proceedings of the London Mathematical Society, vol. 2 (1869), pp. 127–133), who appears to have been the first to generalize the movement of the conchoid mechanism of Nicomedes to the one we are studying. The movement of the mechanism of Nicomedes is obtained by making $a_1 = 0$. Any point upon O_1y_1 then describes a conchoid.

For $a = a_1$ the curves degenerate to cubics. As I have already treated that case (American Journal of Mathematics, vol. 21 (1899), pp. 267–269), it is omitted here.

The fixed centrode c and the moving one c_1 are respectively,

$$a_1^2[x^2 + (y-a)^2] - [x^2 - a(y-a)]^2 = 0,$$
$$a^2[x_1^2 + (y_1 - a_1)^2] - [x_1^2 - a_1(y_1 - a_1)]^2 = 0.$$

The centrode c_1 (Figs. 1, 2) is of the fourth order, symmetrical with respect to the axis of y_1, having a real double point at A_1 and a real tac-node at infinity on

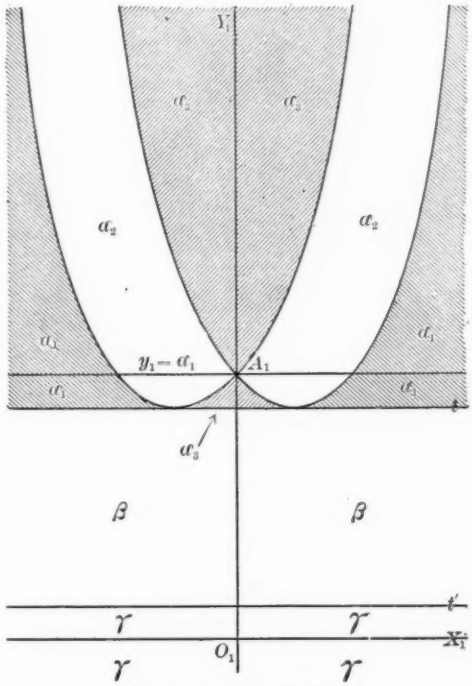

FIG. 2. c_1 for $0 < a < a_1$.

the axis of y_1. The double point at A_1 is a node or a conjugate point according as a_1 is greater or less than a. The line at infinity is tangent at the tac-node and it counts as two double tangents. The other two double tangents are:

$$y_1 = \tfrac{1}{2}(a_1 \pm \sqrt{a_1^2 - a^2}).$$

They are real or imaginary according as $a_1 > a$ or $a_1 < a$. When the double tangents are real the points of contact of that having the positive value of the radical (which we will denote by t) are real, and the points of contact of the other t' are imaginary. When $a > a_1$ the curve has four real inflexions.

When a is zero c_1 degenerates to $x_1^2 = a_1(y_1 - a_1)$ taken twice (Fig. 3). For $a = a_1$ it consists of $x_1 = 0$ taken twice and $x_1^2 = 2a_1y_1 - a_1^2$.

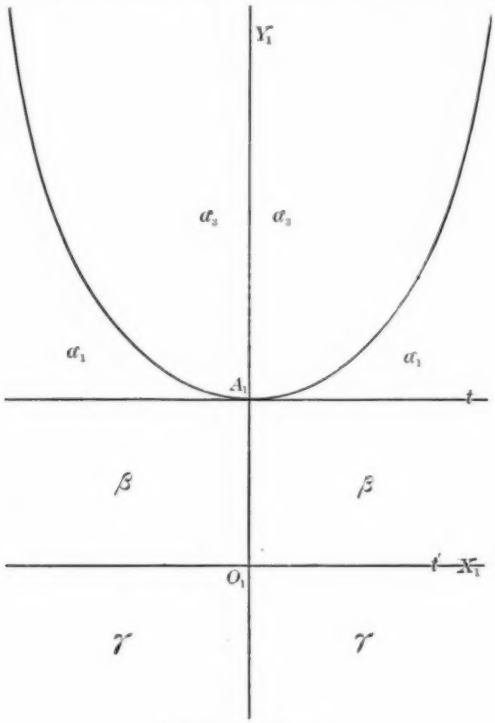

FIG. 3. c_1 for $a = 0$.

The forms of the point loci are shown by the following theorems, the demonstrations of which are omitted.

The locus of any point (x_1, y_1) of σ_1 lies between its tangents:

$$y = \pm \sqrt{x_1^2 + (y_1 - a_1)^2},$$

and has the asymptotes $y = \pm (y_1 - a_1)$ which are tangent to it at its infinitely distant node. The asymptotes are coincident for $y_1 = a_1$ when the curve has a tac-node at infinity. In particular the locus of A_1 consists of $x^2 + (y - a)^2 = 0$ and $y = 0$ each taken twice.

When $a > a_1$ and $y_1 + a_1$ the locus has two branches which come from the direction of positive x from tangency with different asymptotes, pass through the finite part of the plane (where they cross and form a real node N of Fig. 4), and go to infinity in the direction of the negative x-axis each approaching the asymp-

tote from which the other started. The third double point of the locus is in the finite part of the plane, and is a node (M of Fig. 4), cusp, or conjugate point according as (x_1, y_1) is without, on, or between the branches of c_1.

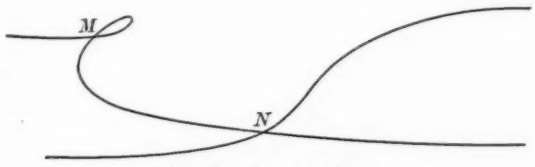

FIG. 4. $a > a_1 ; y_1 \neq a_1$.

When $0 < a < a_1$ and $y_1 \neq a_1$ the locus of (x_1, y_1) consists of two branches. One comes from the direction of positive x from contact with an asymptote into the finite part of the plane and returns to infinity in the direction of positive x and tangency with the other asymptote. The other branch comports itself similarly but comes from the direction of negative x_1 (Fig. 5). The plane σ_1 is

FIG. 5. $a < a_1 ; y_1 \neq a_1$.

divided into three regions α, β, γ by the two double tangents t, t' of c_1 and the line at infinity (Fig. 2). The region α is between t and the line at infinity and contains the real branches of c_1 and the point A_1. The region γ is between t' and the line at infinity, and contains the axis of x_1. The region α is divided into three regions a_1, a_2, a_3 as shown in Fig. 2. The points of a_1 generate curves with two real nodes in addition to that at infinity. They are of two kinds (Fig. 5, I and II) having a node on each branch or two on one according as the trac-

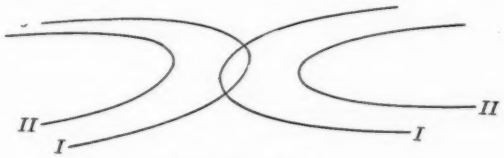

FIG. 6. $a < a_1$.

ing point is not or is between t and $y_1 = a_1$. The loci of points in a_2 have a real node and a real conjugate point, those in a_3 two real conjugate points. On the boundary between a_1 and a_2 points trace curves with one real cusp and one real node, and on the boundary between a_2 and a_3 the loci have one real

cusp and one real conjugate point. The points of t between a_1 and β trace curves which have a tac-node on one branch, which becomes a ramphoid cusp when a point of contact of c_1 and t is taken. Lastly, the portion of t between a_3 and β traces curves with double conjugate points.

The curves of the region β have two non-intersecting branches (II, Fig. 3) and those of γ have two intersecting branches forming two real nodes (I, Fig. 6). On the boundary t' the two branches touch.

When $a = 0$ the two branches of c_1 fall together making the double parabola $x_1^2 = a_1(y_1 - a_1)$ passing through A_1 (Fig. 3). The double tangents t and t' are $y_1 = a_1$ and $y_1 = 0$ respectively. The loci of points of the regions β and γ and the boundary t' have the forms just noted for the case $a \neq 0$, as have also those of a_1 and a_2. The loci of the points of a_1 are of the kind having a node on each branch (Fig. 5, I). The points of c_1 trace curves with two cusps one on each branch, with the exception that the locus of A is degenerate consisting of $x^2 + y^2 = 0$, and $y = 0$ taken twice. The loci of the other points of t have an osenode at infinity.

§4. *Line loci.*

The forms of the unicursal quartic scrolls which are generated by any straight line carried by σ_1 and oblique to it are given by the following theorems.

When l' is parallel to the axis of x_1 the scrolls have three nodal right lines, one of which is a generator at infinity and lies entirely upon the surface. They are of six types as follows:

Type.	Value of $a : a_1$.	Position of l'.	Holgate's	
			Species.	Subform.
I	$a > a_1$		F_5^4	4
II	$a < a_1$	l' lies in the region γ.	F_5^4	5
III	$a < a_1$	l' is t'.	F_6^4	
IV	$a < a_1$	l' is in the region of β.	F_7^4	
V	$a < a_1$	l' is t.	F_6^4	
VI	$a < a_1$	l' is in the region of a.	F_5^4	1

When l' passes through A_1 and is not parallel to the axis of x_1 the surfaces have a nodal hyperbola and intersecting nodal straight line. They are of four types.

Type.	Value of $a : a_1$.	Position of l'.	Holgate's F_3^4 subform.
VII	$a > a_1$		2
VIII	$a < a_1$	l' is in the angle made by the tangents to c_1 at A_1 which contains the axis of y_1.	1
IX	$a < a_1$	l' is tangent to c_1.	4
X	$a < a_1$	l' is in the angle made by the tangents to c_1 at A_1 which does not contain the axis of y_1.	1

When l' neither passes through A_1 nor is parallel to the axis of x_1, the scrolls have a nodal cubical hyperbola and are of the following eleven types:

Type.	Value of $a : a_1$.	Position of l'.	Holgate's $F\frac{1}{4}$ subform.
XI	$a > a_1$	l' intersects c_1 in two real distinct and two imaginary points.	11
XII	$a > a_1$	l' intersects in four real distinct points.	1
XIII	$a > a_1$	l' is tangent at a point of c_1, between which and infinity there is (in one direction) no point of inflexion upon c_1.	4
XIV	$a > a_1$	l' is tangent to c_1 between two inflexions.	2
XV	$a > a_1$	l' is tangent to c_1 at an inflexion.	8
XVI	$a < a_1$	l' intersects c_1 in four imaginary points.	15 or 16*
XVII	$0 < a < a_1$	l' is tangent to c_1 and intersects in two imaginary points.	12 or 14*
XVIII	$0 < a < a_1$	l' intersects c_1 in two real distinct and two imaginary points.	11
XIX	$0 < a < a_1$	l' is tangent to c_1 and intersects it in two real distinct points.	4 or 6*
XX	$a < a_1$	l' intersects c_1 in four real distinct points.	1 or 5*
XXI	$a = 0$	l' is tangent to c_1.	7

In conclusion it is interesting to note that the only plane movements (excluding those trivial ones by which the direction of no carried straight line is changed) by which any carried straight line generates a ruled surface of order not greater than four, are (as far as an examination of the literature of the subject shows) the two of the present paper, those whose centrodes are equal conics,† that of the ellipsograph,‡ and that of rotation about a fixed axis. This suggests the as yet unsolved problem: to make a complete enumeration of such movements.

ITHACA, N. Y., *April* 14, 1900.

* The method followed does not enable one to determine to which of the two subforms the scroll obtained belongs.

† BLAKE, American Journal of Mathematics, vol. 21 (1899), pp. 257–269.

‡ BLAKE, ibid., vol. 22 (1900), pp. 146–153.

THE INVARIANT THEORY OF THE INVERSION GROUP: GEOMETRY UPON A QUADRIC SURFACE*

BY

EDWARD KASNER

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INTRODUCTION.

The extended notion of invariant developed by KLEIN and LIE has become one of the fundamental concepts of mathematical thought: *to every group of transformations there corresponds a geometry, or theory of invariants, dealing with those properties of geometrical or analytical configurations, which are unaltered by the group.** Of the theories which are thus possible few have actually been developed, the most important, of course, being that based upon the total group of linear transformations, i. e., projective geometry or the ordinary theory of forms. The theory of other groups also possessing an algebraic invariant theory has recently received well-deserved consideration, the most prominent investigator being STUDY, who has applied himself to various important subgroups of the general projective group. A remarkable advance in the theory of such groups has been made by MAURER, who has proved that algebraic forms possess a complete system of concomitants not merely with respect to the total group of linear transformations (as had been shown by HILBERT), but also with respect to any subgroup, i. e., with respect to *any* linear group.†

The present paper is concerned with projective geometry upon a non-degenerate quadric surface, or, more specifically, with

(A) *the theory of the algebraic curves upon a proper quadric surface, with respect to those properties which are unaltered by the group of collineations transforming the quadric into itself.*

---

*KLEIN: *Erlanger Programme* (1872); *Höhere Geometrie* (1893).

†Oral communication to the writer by Professor HILBERT; MAURER'S paper in the Münchner Sitzungsberichte was at the time inaccessible. The method employed by MAURER is that developed by HILBERT: *Ueber die Theorie der algebraischen Formen*, Mathematische Annalen, vol. 36, pp. 473–534, 1890, the Ω process of p. 525 being generalized. Whether there are non-linear groups with algebraic invariant theories is still, so far as I know, an unsettled question.

When the quadric reduces to a sphere the group of automorphic collineations has the same effect upon the points of the surface as the group of geometric inversions upon the sphere. By inversion of the sphere into a plane the geometry reduces to

(B) *the inversion geometry of the plane,*

so called since its group is generated by the inversions (transformations by reciprocal radii vectores) of the plane; this group may also be defined as the totality of point transformations which leave the family of circles invariant (MÖBIUS' *Kreisverwandschaft*). The geometries (A) and (B) are equivalent in the general analytic sense which implies only the isomorphism of their groups; furthermore for the invariant methods developed in this paper they are identical.* The nomenclature employed has reference to the one or to the other geometry according as the problem is thus made clearer, but the distinction between them vanishes in the algebraic results.

Abstractly, the fundamental group of either (A) or (B) is a mixed six-parameter group G_6', consisting of two continuous systems of transformations G_6, H_6. These systems are distinguished in the (A) geometry by their effect upon the two sets of generators, the system G leaving each set invariant, while the system H interchanges the sets; and similarly in the (B) geometry by their effect upon the miminal lines or circular points at infinity. The fact that G' is a mixed group, renders its invariant theory in some respects more complicated than that of the continuous subgroup G; for in the G' theory it is not true that the sum (and *a fortiori* an integral function) of concomitants (of course homogeneous) is also a concomitant. It is therefore expedient to consider the G' theory in connection with the G theory.

The principal algebraic methods for treating the geometry (A) or (B) are based upon the following representations of the fundamental group. In the *quaternary method*, tetrahedral or tetracyclic coördinates are employed. The group G' takes the form

$$(1) \qquad G': \qquad x_i' = \sum_k r_{ik} x_k \qquad (i, k = 1, 2, 3, 4),$$

where the transformation coefficients r_{ik} satisfy the conditions which express that the fundamental quadric:

$$(2) \qquad Q \equiv \sum_{i,k} p_{ik} x_i x_k \qquad (p_{ik} = p_{ki};\ \Delta \equiv |p_{ik}| \neq 0)$$

is, except for a factor, transformed into itself—the transformations of G being

*This does not imply that the geometries are necessarily exactly equivalent; but that the treatment of (B) by tetracyclic coördinates coincides with the treatment of (A) by tetrahedral coördinates, and similarly the treatment of (B) by minimal coördinates coincides with the treatment of (A) by generator coördinates.

distinguished by an additional relation (§ 1). In the *double binary method*, parameters $\lambda_1 : \lambda_2$, $\mu_1 : \mu_2$, are introduced in each set of generators or minimal lines, the group taking the form:

$$(3)\quad \begin{aligned} G:&\ \lambda_1' = a\lambda_1 + b\lambda_2,\ \lambda_2' = c\lambda_1 + d\lambda_2;\quad \mu_1' = a\mu_1 + \beta\mu_2,\ \mu_2' = \gamma\mu_1 + \delta\mu_2.\\ H:&\ \lambda_1' = a\mu_1 + b\mu_2,\ \lambda_2' = c\mu_1 + d\mu_2;\quad \mu_1' = a\lambda_1 + \beta\lambda_2,\ \mu_2' = \gamma\lambda_1 + \delta\lambda_2. \end{aligned}$$

In the latter coördinate system, the general algebraic curve $C_{m,n}$ upon the quadric is represented by an equation of the form:

$$\phi(\lambda_1 \overset{m}{:} \lambda_2,\ \mu_1 \overset{n}{:} \mu_2) = 0,$$

where m, n are the partial orders of the curve and indicate the number of points in which it cuts the two sets of generators. The correspondence between the curves and the double binary forms is unique; so that the (A) geometry is adequately represented by

(C) *the theory of double binary forms with respect to independent linear transformations of the variables, and also with respect to the interchange of the variables.*

This theory in connection with the (A) geometry is considered in chapter III.

In the quaternary representation, this unique correspondence between the curves and the forms does not exist. The general curve $C_{m,n}$ (where say $n = m + k$, $k \geqq 0$) requires, for its complete representation, $k + 1$ quaternary forms in point coördinates, each of degree n:

$$f_1, f_2, \cdots f_{k+1}.$$

These $k + 1$ forms are however not unique (unless $m = n = 1$); they may be any $k + 1$ linearly independent members of the linear system:

$$\nu_1 f_1 + \nu_2 f_2 \cdots + \nu_{k+1} f_{k+1} + MQ,$$

where $\nu_1 \cdots \nu_{k+1}$ are constants, and M is an arbitrary form of degree $n - 2$. It is necessary then to distinguish the quaternary theory of the (A) geometry, from

(D) *the theory of quaternary forms with respect to the groups G, G'.*

The foundations for this latter theory have been given by STUDY,* whose starting point is systems of linear forms, and final result the relation of (D) to

(E) *the theory of systems of quaternary forms including a quadric, with respect to the general linear group Γ_{15}.*

*STUDY, *Ueber die Invarianten der projectiven Gruppe einer quadratischen Mannigfaltigkeit von nicht verschwindender Discriminante*, Leipziger Berichte, vol. 49, pp. 443–461, 1897. This paper will be referred to hereafter simply by the name of the author.

In the present paper the theory of the *forms*, i. e. (D), (E), is considered only in so far as the results are necessary for the theory of the *curves*, i. e. (A), (B). The treatment of (D) in § 1 is not to be understood as complete, for in it STUDY's relation between (D) and (E) is assumed. The theory of the curves is taken up in §§ 4, 4′, the former considering the simplest (though most important) class, i. e., the complete intersection or equi-ordinal curves (for which $k = 0$); while the latter, with less development, treats the general curve. The final result of the quaternary method (chapter I) is the reduction of (A) to (E).

The relations between the two methods for treating the (A) or (B) geometry in the case of complete intersection curves—or more explicitly the relations between (C) on the one hand, and (D), (E) on the other—form the subject of chapter IV. The relations are of interest not merely for the (A) geometry, but also for the abstract theory of forms: they lead to principles of transference (Uebertragungsprincip) connecting the theories of quaternary and double binary forms, similar to LINDEMANN's relations between ternary and simple binary forms. * The remaining general theory is contained in chapter VI; the methods there considered may be regarded as variations of the (C) method. In the applications, the (B) terminology is used almost exclusively, the special curves treated being the circles (chapter II) and the cyclics or bicircular quartics (chapter V).

CHAPTER I.

THE QUATERNARY METHOD.

§1. *Quaternary forms with respect to the groups G and G′.*—Each transformation g':

$$x_i' = \sum_k r_{ik} x_k ,$$

of the group G', reproduces the fundamental quadric:

$$Q = p_x^2 \qquad (\Delta = |p_{ik}| \neq 0),$$

except for a factor $D_{g'}$; so that

$$\frac{\sum_{ik} p_{ik} r_{i\alpha} r_{k\beta}}{p_{\alpha\beta}} = D_{g'} \qquad (\alpha, \beta = 1, 2, 3, 4). \tag{1}$$

The transformation *discriminant* $D_{g'}$ is closely related to the transformation *determinant* $\delta = |r_{ik}|$, the former arising in the decomposition of the latter as follows:

* LINDEMANN, *Sur une représentation géométrique des covariantes des formes binaires*, Bulletin Société Mathématique de France, vol. 5, pp. 113–126, 1876; vol. 6, pp. 195–207, 1877. Cf. Mathematische Annalen, vol. 23, pp. 111–142, 1884.

$$|p_{ik}| \cdot |r_{ik}| = |l_{ik}|, \quad \text{where} \quad l_{ik} = \sum_{\alpha} p_{\alpha i} r_{\alpha k};$$

and

$$|l_{ik}| \cdot |r_{ik}| = |m_{ik}|,$$

where

$$m_{ik} = \sum_{\beta} l_{\beta i} r_{\beta \bar{k}} = \sum_{\alpha\beta} p_{\alpha\beta} r_{\alpha i} r_{\beta \bar{k}} = D p_{ik},$$

from (1). Therefore

$$\Delta \delta^2 = \Delta D^4,$$

or

$$\delta = \pm D^2,$$

the positive sign characterising the proper, and the negative the improper transformations; so that we have

$$\delta_g = D_g^2, \quad \delta_h = -D_h^2. \tag{2}$$

In the theory of the concomitants of quaternary forms with respect to the groups G and G',—i. e., of those rational integral functions of the coefficients and variables which are unaltered (except for a factor independent of the coefficients and variables) by transformations g and g', respectively,—the preceding formulæ are fundamental. Considering for simplicity the invariants of a single form

$$f = a_x^n,$$

we have for a G invariant I,

$$I(a_g) = \phi_g(r) I(a), \tag{3}$$

and for a G' invariant

$$I(a_g) = \phi_g(r) I(a), \quad I(a_h) = \phi_h(r) I(a), \tag{4}$$

where a_g, a_h represent the coefficients of the transformed form, and ϕ is independent of the a's. The inverse of g and h being also members of G and H,

$$I(a) = \phi_g\left(\frac{R}{\delta}\right) I(a_g), \quad I(a) = \phi_h\left(\frac{R}{\delta}\right) I(a),$$

where R_{ik} is the minor of r_{ik} in $\delta = |r_{ik}|$; therefore in both cases

$$\phi(r)\phi\left(\frac{R}{\delta}\right) = 1 \quad \text{or} \quad \phi(r)\phi(R) = \delta^{\lambda} = \pm D^{2\lambda},$$

from (2). The discriminant D being irreducible, $\phi(r)$ is of the form cD^{μ}, say

$$\phi_g = c_1 D_g^{\mu}, \quad \phi_h = c_2 D_h^{\mu}.$$

The laws of combination of the transformations g and h give

$$c_1^2 = c_1, \; c_1 c_2 = c_2, \; c_2^2 = c_1; \quad \text{or} \quad c_1 = 1, \; c_2 = \pm 1.$$

For a G invariant the transformation factor is a power of the discriminant D^{μ}. For a G' invariant the factor produced by the transformations g is D^{μ}, while that produced by h may be either D^{μ} or $-D^{\mu}$, thus creating a division of the G' invariants into even and odd.

The theory of G and G' invariants is closely related to the ordinary theory of forms, the connection being expressed in the following theorems.

1°.—*Every projective invariant T, of*

$$Q = p_x^2, \quad f = a_x^n,$$

is a G' invariant of f.

Representing the general transformation of the projective group Γ_{15} by γ, we have by assumption,

$$T(a_\gamma, p_\gamma) = \delta_\gamma^\lambda T(a, p),$$

where λ is the weight of T. Therefore

$$T(a_{g'}, p_{g'}) = T(a_{g'}, D_{g'}p) = \delta_{g'}^\lambda T(a, p),$$

$$T(a_{g'}, p) = \frac{\delta_{g'}^\lambda}{D_{g'}^\mu} T(a, p) = \pm D_{g'}^{2\lambda-\mu} T(ap),$$

μ being the degree of T in the p's. Considered as a G' invariant, T is even or odd according as its weight λ is even or odd.

2°.—*Every sum $\sum \phi_i(p) T_i$, where the ϕ's are arbitrary functions and the T's are all of the same degree in the a's, is a G invariant; if, in addition, the weights of the T's are either all even or all odd, the sum is a G' invariant.*

This follows from 1° and from the fact that the exponent ν, of the transformation factor $\pm D^\nu$ due to each term of the sum, depends only upon the degree in the a's, while the sign depends only upon whether the weight is even or odd.

The converse of 2° has in substance been proved by STUDY* in the paper already cited. This converse is

3°.—*Every G invariant can be expressed in the form $\sum \phi_i(p) T_i$, the T's being all of the same degree in the a's; in addition, for a G invariant the weights of the T's will be either all even or all odd.*

Relations between the complete systems of concomitants for the (D) and (E) theories of the introduction, follow directly from these theorems. *A complete T system*, i. e., a system $T_1, T_2, \cdots$ such that every T can be written $T = R(T_1, T_2, \cdots)$, where R is a rational integral function with numerical coefficients,—*is also a complete G system*, i. e., every G can be written $G = R_p(T_1, T_2, \cdots)$, where R_p is a rational integral function with coefficients which may involve the p's. More generally, a system $T_1, T_2, \cdots$ such that every T can be written $T = R_p(T_1, T_2, \cdots)$, is a complete G system; conversely, from

* Leipziger Berichte, vol. 49, p. 458.

every complete G system a system of T's can be derived with the property expressed by $T = R_\rho(T_1, T_2, \cdots)$.

§ 2. *Systems of quaternary forms containing a quadric, with respect to the general projective group* Γ_{15}.—The simplicity of the theory of systems of forms containing a quadric, arises from the possibility of a reduction, peculiar to them, of contragrediency to cogrediency.*

In the first place *the original system* S, *containing a quadric and forms in any number of point and plane coördinates, may be replaced by another system* S', *containing only point coördinates.* For simplicity consider only one set of point coördinates x, and one of plane coördinates u. Let the system S be

$$S: \qquad Q = p_x^2, \ f = a_x^n, \ \phi = u_\alpha^m, \ \chi = b_x^r u_\beta^s.$$

By the substitution $$u_i = p_i p_y,$$
this becomes

$$S': \qquad \begin{gathered} Q, \ f, \ \bar{\phi} = p_\alpha p'_\alpha \cdots p_\alpha^{(m-1)} p_y p'_y \cdots p_y^{(m-1)} = A_y^m, \\ \bar{\chi} = p_\beta \cdots p_\beta^{(s-1)} p_y \cdots p_y^{(s-1)} b_x^r = b_x^r B_y^s; \end{gathered}$$

which in turn, by the substitution

$$y_i = P_i u_P,$$

becomes

$$S_1: \qquad \begin{gathered} Q, \ f, \ \bar{\bar{\phi}} = A_P \cdots A_{P^{(m-1)}} u_P \cdots u_{P^{(m-1)}} = \Delta^m \phi, \\ \bar{\bar{\chi}} = b_x^r B_P \cdots B_{P^{(s-1)}} u_P \cdots u_{P^{(s-1)}} = \Delta^s \chi. \end{gathered}$$

The system S' is composed of concomitants of S, and the system S_1, of concomitants of S'; therefore every concomitant of S' is a concomitant of S, and every concomitant of S_1 is a concomitant of S'. Again, since S_1, except for powers of Δ, coincides with S, every concomitant of S, multiplied by a suitable power of Δ, is a concomitant of S_1 and therefore also of S'. If then we regard the discriminant Δ, of the quadric Q, as a number, the systems S and S' are equivalent.

In the second place, *if again we regard* Δ *as a number, it is sufficient, in the study of complete form systems, to consider only in- and covariants.* For, disregarding powers of Δ, by the substitution

$$u_i = p_i p_y,$$

the totality of contravariants passes over into the totality of covariants; and mixed concomitants (i. e., those involving both point and plane coördinates) become covariants in two or more sets of congredient variables.

* STUDY, pp. 458, 459 gives this reduction for the (D) theory; in the above the starting point is the (E) theory. Cf. the introduction.

§ 3. *The apolar surface.*—Consider the curve of intersection of the quadric $Q = 0$, and the n-ic $f = 0$. Of the totality of n-ic surfaces which pass through the curve, and which are therefore represented by

$$\lambda f + MQ = 0 \tag{1}$$

(λ being a constant and M an arbitrary form of degree $n-2$), there is one which is of special importance in the theory of the curve, i. e., that one which is apolar to Q considered as a class quadric,

$$\overline{Q} = u_P^2 = \tfrac{1}{6}(pp'p''u)^2.$$

A surface is apolar to $\overline{Q}$ when all its polar quadrics are harmonically circumscribed with respect to $\overline{Q}$, i. e., when every polar quadric is circumscribed about an infinity of tetrahedra which are self-polar with respect to $\overline{Q}$.* The covariant

$$S = a_P^2 a_x^{n-2} = \sum P_{ik} \frac{\partial^2 f}{\partial x_i \partial x_k},$$

equated to zero, gives the locus of points whose polar quadrics are harmonic to $\overline{Q}$: its identical vanishing is therefore the necessary and sufficient condition for the apolarity of f and $\overline{Q}$.

If $F = \lambda f + MQ$ is apolar to $\overline{Q}$,

$$\Omega F = 0, \quad \text{where } \Omega = \sum P_{ik} \frac{\partial^2}{\partial x_i \partial x_k}. \tag{3}$$

It is necessary now to calculate $\Omega(MQ)$. We have

$$\frac{\partial^2}{\partial x_i \partial x_k}(MQ) = Q \frac{\partial^2 M}{\partial x_i \partial x_k} + M \frac{\partial^2 Q}{\partial x_i \partial x_k} + \frac{\partial Q}{\partial x_i}\frac{\partial M}{\partial x_k} + \frac{\partial Q}{\partial x_k}\frac{\partial M}{\partial x_i},$$

or

$$\Omega(MQ) = Q\Omega M + M\Omega Q + 2\sum P_{ik} \frac{\partial Q}{\partial x_i}\frac{\partial M}{\partial x_k};$$

but

$$\Omega Q = 2\sum P_{ik}p_{ik} = 8\Delta,$$

and

$$\sum P_{ik} \frac{\partial M}{\partial x_k}\frac{\partial Q}{\partial x_i} = 2\sum_{k,l} x_l \frac{\partial M}{\partial x_k} \sum_i P_{ik}p_{il}$$

$$= 2\Delta \sum x_i \frac{\partial M}{\partial x_i}$$

$$= 2(n-2)\Delta M.$$

* REYE, *Ueber Algebraischen Flächen die zu einander Apolar sind*, Crelle, vol. 79, pp. 159–175, 1874.

Introducing these values, we have

$$\Omega(MQ) = Q\Omega M + 4n\Delta M, \tag{4}$$

and by repeated application,

$$\Omega^2(MQ) = Q\Omega^2 M + 8(n-1)\Delta\Omega M,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\Omega^h(MQ) = Q\Omega^h M + 4h(n-h+1)\Delta\Omega^{h-1}M.$$

From (2) and (3),

$$\begin{aligned} &F \equiv \lambda f + MQ, \\ &\Omega f + Q\Omega M + 4n\Delta M \equiv 0, \\ &\Omega^2 f + Q\Omega^2 M + 8(n-1)\Delta\Omega M \equiv 0, \\ &\cdots\cdots\cdots \\ &\Omega^h f + Q\Omega^h M + 4h(n-h+1)\Delta\Omega^{h-1}M \equiv 0. \end{aligned} \tag{5}$$

Multiplying these equations (5) by

$$1, \ -\frac{Q}{4\cdot n}, \ \frac{Q^2}{4\cdot 8\cdot n(n-1)}, \ \cdots,$$

respectively, adding and putting $\lambda = \Delta^\nu$ (where ν is the greatest integer contained in $n/2$), we have the required determination,

$$\begin{aligned} F = \Delta^\nu f - \frac{\Delta^{\nu-1}}{1\cdot n}\frac{Q}{4}\Omega f + \frac{\Delta^{\nu-2}}{1\cdot 2\cdot n(n-1)}\frac{Q^2}{4^2}\Omega^2 f - \cdots \\ + (-1)^\nu \frac{\Delta^\circ}{\nu!\cdot n(n-1)\cdots(n-\nu+1)}\frac{Q^\nu}{4^\nu}\Omega^\nu f. \end{aligned} \tag{6}$$

From its definition F is a covariant of f and Q; to put this in evidence we may express it symbolically as follows: *

$$\begin{aligned} 4^\nu F = \Delta^\nu a_x^n - \frac{n(n-1)}{1\cdot n}\Delta^{\nu-1}Qa_P^2 a_x^{n-2} + \frac{n\cdots(n-3)}{1\cdot 2\cdot n(n-1)}\Delta^{\nu-2}Q^2 a_P^2 a_{P'}^2 a_x^{n-4} - \cdots \\ + (-1)^\nu \frac{n\cdots(n-2\nu+1)}{1\cdots\nu\cdot n\cdots(n-\nu+1)}\Delta^\circ Q^\nu a_P^2 a_{P'}^2 \cdots a_{P^{(\nu-1)}}^2 a_x^{n-2\nu}. \end{aligned} \tag{7}$$

This determination proves that there is one and only one surface $F = 0$, which passes through the curve and is apolar to the quadric. †

* The corresponding ternary (and n dimensional) problem can be treated in the same way; it has been considered by a different method by LINDEMANN, Bulletin Société Mathématique de France, vol. 6, pp. 195-207, 1877.

† This and a more general theorem is given by STUDY, *Ueber quadratischen Formen und Linien complexe*, Leipziger Berichte, vol. 49, p. 174, 1890.

§ 4. *Equi-ordinal* * *or complete intersection curves on a quadric with reference to G and G′.*—The theory of the curve

$$f = 0, \quad Q = 0,$$

on the fundamental quadric, with reference to G and G' (or, what is equivalent, the inversion theory of the curve whose equation in tetracyclic coördinates connected by the quadratic equation $Q = 0$, is $f = 0$), is obviously not identical with the theory of the form f, since the latter is not completely determined by the curve. It is necessary then to *define* what is meant by a concomitant of the curve:

A G (or G') concomitant of the CURVE $f = 0$, *is a "special"* † *G (or G') concomitant of the* FORM $f = 0$, i. e., *a concomitant I_f with the property expressed by the equation:*

$$I_{\lambda f + MQ} = \psi I_f,$$

where ψ is independent of the coefficients and variables involved in I_f.

1°. The apolar form:

$$F = \Delta^v f - \frac{\Delta^{v-1}}{1 \cdot u} \frac{Q}{4} \Omega f + \cdots = \Delta^v f + \bar{M} Q$$

is a "special" covariant of f.

For from (6) § 3, we have $F(\lambda f + MQ) = \lambda F$.

2°. Every concomitant of F is a "special" concomitant of f. Since I is a concomitant of F it is also a concomitant of f:

$$I_F = \bar{I}_f.$$

Again, $\bar{I}$ is "special." For, putting $f' = \lambda f + MQ$,

$$\bar{I}_{f'} = I_{F'} = I_{\lambda F} = \lambda^\kappa I_F = \lambda^\kappa \bar{I}_f.$$

3°. Every "special" concomitant of f is a concomitant of F.

By assumption,

$$I_{\lambda f + MQ} = \psi I_f;$$

therefore

$$I_F = \psi' I_f,$$

$$I_f = \frac{I_F}{\psi'} = I'_F.$$

* *Cf.* § 10; the general curves are considered in § 4′.

[† September 28, 1900. *Cf.* H. S. WHITE'S definition of *semi-combinant* in his paper, *Semi-combinants as concomitants of affiliants*, American Journal of Mathematics, vol. 17, pp. 234–265, 1895. The form F of the text is a special *affiliant*. The theorems 1°, 2°, 3°, relating to G concomitants, together with their proofs, are almost identical with corresponding theorems for Γ semi-combinants given in the same paper; in fact the former may be regarded as merely special cases of the latter.]

From these results we obtain the theorem:

The G (or G') concomitants of the curve $f=0$ coincide with the G (or G') concomitants of the apolar form F.*†

This theorem, in connection with the results of § 1, reduces the problem of finding the G or G' concomitants of the curve, to the ordinary theory of forms.

§ 4′. *The general curve.*—In the preceding section only those curves were considered which can be defined by one quaternary form in point coördinates; or, in the nomenclature of § 10, only equi-ordinal curves. A purely quaternary method for treating *all* algebraic curves is suggested in the following.

The general algebraic curve C_{mn}, on the fundamental quadric Q, whose partial orders (§ 10) are m and n (where $n = m + k$, $k \geqq 0$), may be defined by $k+1$ quaternary forms of order n,

$$f_1, f_2, \cdots f_{k+1},$$

which represent $k+1$ linearly independent n-ic surfaces through the curve. The totality of n-ic surfaces passing through the curve forms the linear system:

$$S = \nu_1 f_1 + \nu_2 f_2 + \cdots + \nu_{k+1} f_{k+1} + MQ,$$

where $\nu_1 \cdots \nu_{k+1}$ are arbitrary constants and M is an arbitrary form of order $n-2$. The definition given in § 4 may be thus generalized:‡

* By the *curve* $f=0$, is meant either the *plane* curve whose equation in tetracyclic coördinates connected by the identity $Q=0$, is $f=0$; or the *space* curve upon the quadric $Q=0$, cut out by the surface $f=0$.

† The apolar form F corresponding to the general curve $f=0$, is not the general form of degree n: the relation of F to Q causes a certain covariant of degree $10(n-4)$ to vanish identically. This covariant may be expressed as a determinant of the tenth order:

$$C = \begin{vmatrix} (1111) & (1112) & (1113) & (1114) & (1122) & (1123) & (1124) & (1133) & (1134) & (1144) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (4411) & (4412) & (4413) & (4414) & (4422) & (4423) & (4424) & (4433) & (4434) & (4444) \end{vmatrix},$$

where

$$(iklm) = \frac{1}{n(n-1)(n-2)(n-3)} \frac{\partial^4 f}{\partial x_i \partial x_k \partial x_l \partial x_m}.$$

The symbolic expression for this covariant given by LINDEMANN, Mathematische Annalen, vol. 23, pp. 136, 137, is incorrect; the correct expression requires the solution of the hitherto unsolved problem of the symbolic quadric through ten points.

‡ A second definition applicable to all curves, but which is not a generalization of that given in § 4, is obtained by starting from the tangential equation

$$\phi \equiv u_t^{2mn} = 0,$$

of the curve C_{mn}; or, what is equivalent, from the reciprocal of ϕ with respect to Q,

$$\bar{\phi} \equiv p_t p_t' \cdots p_t^{(2mn-1)} p_x p_x' \cdots p_x^{(2mn-1)}.$$

(The surface $\bar{\phi}=0$ cuts the fundamental quadric Q in the curve C_{mn} counted twice, and in those generators of Q which are tangent to C_{mn}). This representation suggests the definition: *The G (or G') concomitants of the curve C_{mn} are the G (or G') concomitants of the quaternary form ϕ; or, (what is equivalent except for powers of Δ), of the form $\bar{\phi}$.* Concomitants as here defined are also concomitants as defined in the text, but the converse is not true.

A G (or G') concomitant of the curve $C_{m,n}$ is a G (or G') concomitant I of the forms $f_1, f_2, \cdots f_{k+1}$, which possesses the property defined by the equation:

$$I(\sum v'f + M'Q,\ \sum v''f + M''Q,\ \cdots) = \psi I(f_1, f_2, \cdots),$$

where ψ is independent of the coefficients and variables involved in $I(f_1, f_2 \cdots)$.

When the curve is equi-ordinal, this definition reduces to that given in § 4. The fundamental theorem for the concomitants of the general curve is:

The G (or G') concomitants of the curve $C_{m,n}$ are the G (or G') combinants (that is, concomitants with the combinant property) of the apolar forms $F_1, \cdots F_{k+1}$, corresponding to $f_1, \cdots f_{k+1}$.

CHAPTER II.

Concomitants of Circles.

§ 5. *Quaternary system.*—Let the equations of any number of circles, in tetracyclic coördinates connected by the identity

$$Q \equiv \sum p_{ik} x_i x_k \equiv 0,$$

be

$$\begin{aligned} f_1 &= A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = 0, \\ f_2 &= B_1x_1 + B_2x_2 + B_3x_3 + B_4x_4 = 0, \\ f_3 &= C_1x_1 + C_2x_2 + C_3x_3 + C_4x_4 = 0, \quad \text{etc.} \end{aligned} \tag{1}$$

The complete system of inversion concomitants may be derived, by the method of the preceding chapter,* from the projective study of the form system

$$S: \qquad Q = p_x^2, \quad f_1 = A_x, \quad f_2 = B_x, \quad \cdots,$$

the discriminant (of the fundamental quadric Q)

$$\Delta = \tfrac{1}{24}(pp'p''p''')^2$$

being regarded as a mere number. With this last proviso, the system S is equivalent to the system

$$S': \qquad \overline{Q} = u_P^2 = \tfrac{1}{6}(pp'p''u)^2, \quad f_1 = A_x, \quad f_2 = B_x, \quad \cdots,$$

the proof being similar to that given in § 2 for the equivalence of the systems there denoted by the same letters S and S'.

Invariants of the system S' may be considered as concomitants in several sets of cogredient variables A, B, $\cdots$ of the single form Q: we may there-

* In the case of circles, the concomitants of the forms (§ 1), and the concomitants of the curves (§ 4), coincide, so that no reduction by apolarity is necessary (from § 3, $F = \Delta \cdot f$).

fore apply a method of CLEBSCH,* by which the concomitants in several sets of cogredient variables are deduced from identical concomitants and polars of concomitants in one set of variables. The invariants are thus found to be of only two types:

$$A_P B_P \text{ (including the type } A_P^2),$$

and

$$(ABCD).$$

The contravariants, being invariants of S' and an additional circle, are

$$u_P^2,\ A_P u_P,\ (ABCD);$$

and the covariants, by the second principle of § 2, are

$$A_x,\ (ABCp)p_x.$$

Collecting the preceding results one concludes:

Every G or G' concomitant of the n linear forms $f_1, f_2, \cdots, f_n$, that is, every inversion concomitant of the n corresponding circles, is a rational integral function (the coefficients of which may involve the quantities p_{ik}) of the following concomitants:†

$\frac{n(n+1)}{2}$	*invariants of the type*	$I_{12} = A_P B_P,$
$\binom{n}{4}$	" " " "	$K_{1234} = (ABCD),$
n	*covariants* " " "	$f_1 = A_x,$
$\binom{n}{3}$	" " " "	$g_{123} = (ABCp)p_x,$
n	*contravariants of the type*	$F_1 = A_P u_P,$
$\binom{n}{3}$	" " " "	$G_{123} = (ABCu),$
and the identical contravariant		$Q = u_P^2.$

The non-symbolic expressions for these forms, in the case of the orthogonal system of tetracyclic coördinates ($Q \equiv \sum x^2$), are

$$f_1 = \sum A_i x_i,\quad F_1 = \sum A_i u_i,\quad I_{12} = \sum A_i B_i,$$
$$K_{1234} = (ABCD),\quad g_{123} = (ABCx),\quad G_{123} = (ABCu).$$

*CLEBSCH, *Ueber ein fundamentale Aufgabe der Invariantentheorie*, Göttingen Abhandlungen, vol. 17, p. 39, 1872.

† These (or rather equivalent) forms are determined directly by STUDY (p. 443 ff.), i. e., without passing to the ordinary theory of invariants; the same is true of the quaternary syzygies of § 7.

§ 6. *Binary system.*—In minimal coördinates* λ, μ, the equation of a circle is

$$a_{11}\lambda\mu + a_{12}\lambda + a_{21}\mu + a_{22} = 0,$$

or in homogeneous form,

$$f_1 \equiv a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 = 0.$$

A circle is therefore represented by a binary bilinear form with non-cogredient variables. Such forms have been studied by PEANO.† If the n circles be represented by the symbols of their bilinear forms,

$$f_1 = a_\lambda a_\mu, \quad f_2 = b_\lambda \beta_\mu, \quad f_3 = c_\lambda \gamma_\mu, \quad \cdots,$$

their complete binary system is composed of the following forms:

(1)
$$\begin{array}{lll}
n & \text{of type } f_1 & = a_\lambda a_\mu, \\
n & \text{“} \quad F_1 & = (ar)(a\rho), \\
\dfrac{n(n+1)}{2} & \text{“} \quad I_{12} & = (ab)(a\beta), \\
\binom{n}{4} & \text{“} \quad K_{1234} & = (ac)(bd)(a\beta)(\gamma\delta) - (ab)(cd)(a\gamma)(\beta\delta), \\
\binom{n}{3} & \text{“} \quad g_{123} & = (ac)(a\beta)b_\lambda\gamma_\mu - (ab)(a\gamma)c_\lambda\beta_\mu, \\
\binom{n}{3} & \text{“} \quad G_{123} & = (ac)(a\beta)(br)(\gamma\rho) - (ab)(a\gamma)(cr)(\beta\rho), \\
\binom{n}{2} & \text{“} \quad L_{12} & = (a\beta)a_\lambda b_\mu, \\
\binom{n}{2} & \text{“} \quad M_{12} & = (ab)a_\mu\beta_\mu, \\
1 & \text{“} \quad Q & = (rr')(\rho\rho'),
\end{array}$$

where the variable circle occurring in the contravariants is $r_{11}\lambda_1\mu_1 + r_{12}\lambda_1\mu_2 + r_{21}\lambda_2\mu_1 + r_{22}\lambda_2\mu_2 = r_\lambda\rho_\mu = r'_\lambda\rho'_\mu$. This system may be deduced by PEANO's direct method; it may also be obtained (except for L and M) by a transformation of the quaternary results of § 5. (Cf. chapter IV.)

* These are discussed in § 10. The simplest element in these coördinates is not the circle, but the minimal line, whose equation is of the form $a_1\lambda_1 + a_2\lambda_2 = 0$, or $a_1\mu_1 + a_2\mu_2 = 0$. The complete system of any number of such lines $a_\lambda, b_\lambda, \cdots, a_\mu, \beta_\mu \cdots$, consists of the forms themselves and of the invariants (ab), $(a\beta)$, etc.

† PEANO, *Formazioni invariantive della correspondenza*, Giornale di Matematiche, vol. 20, pp. 81-88, 1882.

The non-symbolic values of the concomitants (1) are

$$f_1 = a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2,$$
$$F_1 = a_{11}r_{22} - a_{12}r_{21} - a_{21}r_{12} + a_{22}r_{22},$$
$$I_{11} = 2a_{11}a_{22} - 2a_{12}a_{21}, \quad I_{12} = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11},$$

$$K_{1234} = \begin{vmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \\ d_{11} & d_{12} & d_{21} & d_{22} \end{vmatrix}, \quad G_{123} = \begin{vmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \\ r_{11} & r_{12} & r_{21} & r_{22} \end{vmatrix},$$

$$g_{123} = \begin{vmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \\ \lambda_2\mu_2 & -\lambda_2\mu_1 & -\lambda_1\mu_2 & \lambda_2\mu_2 \end{vmatrix}, \quad Q = 2r_{11}r_{22} - 2r_{12}r_{21},$$

$$L_{12} = (a_{11}b_{12} - a_{12}b_{11})\lambda_1^2 + (a_{11}b_{22} - a_{12}b_{21} + a_{21}b_{12} - a_{22}b_{11})\lambda_1\lambda_2 + (a_{21}b_{22} - a_{22}b_{21})\lambda_2^2,$$
$$M_{12} = (a_{11}b_{21} - a_{21}b_{11})\mu_1^2 + (a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11})\mu_1\mu_2 + (a_{12}b_{22} - a_{22}b_{12})\mu_2^2.$$

§ 7. *Syzygies in the preceding systems.*—A series of simple considerations shows that the systems given in §§ 1, 2, are not only complete, but also irreducible; however the members of the systems are not independent. For the quaternary system the relations among the invariants are*

$$(1) \qquad \Delta^3 K_{\alpha\beta\gamma\delta} K_{\alpha'\beta'\gamma'\delta'} = \begin{vmatrix} I_{\alpha\alpha'} & I_{\alpha\beta'} & I_{\alpha\gamma'} & I_{\alpha\delta'} \\ I_{\beta\alpha'} & I_{\beta\beta'} & I_{\beta\gamma'} & I_{\beta\delta'} \\ I_{\gamma\alpha'} & I_{\gamma\beta'} & I_{\gamma\gamma'} & I_{\gamma\delta'} \\ I_{\delta\alpha'} & I_{\delta\beta'} & I_{\delta\gamma'} & I_{\delta\delta'} \end{vmatrix},$$

$$(2) \qquad I_{i\alpha} K_{\beta\gamma\delta\varepsilon} + I_{i\beta} K_{\gamma\delta\varepsilon\alpha} + I_{i\gamma} K_{\delta\varepsilon\alpha\beta} + I_{i\delta} K_{\varepsilon\alpha\beta\gamma} + I_{i\varepsilon} K_{\alpha\beta\gamma\delta} = 0.$$

The relation (1) is proved by decomposing the right hand member, after substituting the symbolic values given in §5, as follows:

$$\begin{vmatrix} A_P A'_P & A_P B'_P & A_P C'_P & A_P D'_P \\ B_{P'} A'_{P'} & B_{P'} B'_{P'} & B_{P'} C'_{P'} & B_{P'} D'_{P'} \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \end{vmatrix}$$

*STUDY, l. c., p. 444, gives relations equivalent to (1) and (2), and also proves that they form the complete system of syzygies.

$$= A_P B_{P'} C_{P''} D_{P'''} \begin{vmatrix} A'_P & B'_P & C'_P & D'_P \\ A'_{P'} & B'_{P'} & C'_{P'} & D'_{P'} \\ . & . & . & . \\ . & . & . & . \end{vmatrix}$$

$$= \tfrac{1}{24} \begin{vmatrix} A_P & B_P & C_P & D_P \\ A_{P'} & B_{P'} & C_{P'} & D_{P'} \\ . & . & . & . \\ . & . & . & . \end{vmatrix} \cdot \begin{vmatrix} A'_P & B'_P & C'_P & D'_P \\ A'_{P'} & B'_{P'} & C'_{P'} & D'_{P'} \\ . & . & . & . \\ . & . & . & . \end{vmatrix}$$

$$= \tfrac{1}{24}(ABCD)(A'B'C'D')(PP'P''P''')^2$$

$$= \Delta^3(ABCD)(A'B'C'D').$$

The relation (2) is proved by rewriting the first member as follows:

$$F_P \begin{vmatrix} A_P & B_P & C_P & D_P & E_P \\ A_1 & B_1 & C_1 & D_1 & E_1 \\ . & . & . & . & . \\ . & . & . & . & . \\ A_4 & B_4 & C_4 & D_4 & E_4 \end{vmatrix}.$$

The relations among the remaining forms of the quaternary system are of the same two types. Only a few, which are to be employed later, will here be written out.

In (1) let f_δ and $f_{\delta'}$ be the same degenerate circle, i. e., point circle; then

$$\Delta^3 g_{\alpha\beta\gamma} g_{\alpha'\beta'\gamma'} = \begin{vmatrix} I_{\alpha\alpha'} & I_{\alpha\beta'} & I_{\alpha\gamma'} & f_\alpha \\ I_{\beta\alpha'} & I_{\beta\beta'} & I_{\beta\gamma'} & f_\beta \\ I_{\gamma\alpha'} & I_{\gamma\beta'} & I_{\gamma\gamma'} & f_\gamma \\ f_{\alpha'} & f_{\beta'} & f_{\gamma'} & 0 \end{vmatrix}. \tag{3}$$

Again, let f_i in (2) become a point; then

$$f_\alpha K_{\beta\gamma\delta\epsilon} + f_\beta K_{\gamma\delta\epsilon\alpha} + f_\gamma K_{\delta\epsilon\alpha\beta} + f_\delta K_{\epsilon\alpha\beta\gamma} + f_\epsilon K_{\alpha\beta\gamma\delta} = 0. \tag{4}$$

Another important syzygy among the invariants is

$$\begin{vmatrix} I_{\alpha\alpha'} & I_{\alpha\beta'} & I_{\alpha\gamma'} & I_{\alpha\delta'} & I_{\alpha\epsilon'} \\ I_{\beta\alpha'} & I_{\beta\beta'} & I_{\beta\gamma'} & I_{\beta\delta'} & I_{\beta\epsilon'} \\ I_{\gamma\alpha'} & I_{\gamma\beta'} & I_{\gamma\gamma'} & I_{\gamma\delta'} & I_{\gamma\epsilon'} \\ I_{\delta\alpha'} & I_{\delta\beta'} & I_{\delta\gamma'} & I_{\delta\delta'} & I_{\delta\epsilon'} \\ I_{\epsilon\alpha'} & I_{\epsilon\beta'} & I_{\epsilon\gamma'} & I_{\epsilon\delta'} & I_{\epsilon\epsilon'} \end{vmatrix} = 0, \tag{5}$$

which is however not distinct from (1) and (2), for the above determinant may be written:

$$\Delta^3 K_{\alpha'\beta'\gamma'\delta'}(I_{\epsilon'\alpha}K_{\beta\gamma\delta\epsilon} + I_{\epsilon'\beta}K_{\gamma\delta\epsilon\alpha} + \cdots).$$

The above relations apply also to the binary system, if in them we put $\Delta = 1$; in addition,*

$$-2L_{\alpha\beta}M_{\gamma\delta} = I_{\alpha\gamma}f_\beta f_\delta - I_{\alpha\delta}f_\beta f_\gamma - I_{\beta\gamma}f_\alpha f_\delta + I_{\beta\delta}f_\alpha f_\gamma + f_\gamma g_{\alpha\delta\beta} - f_\alpha g_{\beta\gamma\delta}.$$

§ 8. *Geometric interpretations.*†—Two systems of concomitants of circles have been given in §§ 6, 7. The first applies also to a quadric and a set of planes, the second also to a set of binary homographies in which the carriers are not regarded as superposed: for each of these points of view there is a corresponding geometric interpretation which will not, however, be given here.

If then only circles in the plane be considered, the interpretations of the vanishing of the concomitants are as follows:

$f_1 = 0$, represents a circle; and

$F_1 = 0$, the linear complex of circles orthogonal to f_1.

$I_{11} = 0$, is the condition for the degeneration of the circle f; and

$I_{12} = 0$, is the condition for the orthogonality of f_1 and f_2.

$K_{1234} = 0$, is the condition that the four circles f_1, f_2, f_3, f_4 belong to a linear system, i. e., have a common orthogonal circle.

$g_{123} = 0$, represents the circle orthogonal to f_1, f_2, f_3; and

$G_{123} = 0$, represents the linear complex orthogonal to g_{123}.

$\bar{Q} = 0$, represents the quadratic complex consisting of all the degenerate circles of the plane.

The preceding forms are common to both systems; in the binary system we have, in addition, L_{12}, M_{12}, which represent the pairs of minimal lines of each series through the intersections of f_1 and f_2.

All invariant relations of circles must be representable through the above concomitants; some examples will now be considered.

* PEANO, l. c.

† On the geometry of circles see STUDY, *Das Apollonische Problem*, Mathematische Annalen, vol. 49, 1897.

MÖBIUS, *Kreisverwandschaft*, Werke, vol. II, 1852.

LORIA, *Geometria della sfera*, Memorie Accademia Torino, vol. 36, 1885.

LACHLAN, Philosophical Transactions of the Royal Society, 1886.

The condition for the tangency of two circles f_1, f_2 is

$$\begin{vmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{vmatrix} = 0;$$

and the condition that three circles f_1, f_2, f_3, have a common point is

$$\begin{vmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{vmatrix} = 0.$$

These are combinants applying equally to any two or three members of the linear systems $t_1 f_1 + t_2 f_2$, $t_1 f_1 + t_2 f_2 + t_3 f_3$, respectively.

The circles cutting f_1, f_2 in four points whose anharmonic ratio is a, are given by

$$\left(\frac{a+1}{a-1}\right)^2 = \frac{(I_{12}\sigma' - F_1 F_2)^2}{(I_{11}\sigma' - F_1^2)(I_{22}\sigma' - F_2^2)},$$

the harmonic circles (i. e., $a = -1$) being

$$I_{12}\sigma' - F_1 F_2 = 0.$$

As an application, let it be required to find the equation, in invariant form, of the cyclic curve* through eight given points †. The circle through the points 1, 2, 3, is $g_{123} = 0$; the system of circles through six of the points is therefore:

$$t_1 g_{123} g_{456} + t_2 g_{124} g_{356} + t_3 g_{125} g_{346} = 0.$$

If we impose the conditions for the curve going through the remaining two points, and eliminate t_1, t_2, t_3, the required cyclic is found to be

$$\begin{vmatrix} g_{123}\, g_{456} & g_{124}\, g_{356} & g_{125}\, g_{346} \\ K_{1237} K_{4567} & K_{1247} K_{3567} & K_{1257} K_{3467} \\ K_{1238} K_{4568} & K_{1248} K_{3568} & K_{1258} K_{3468} \end{vmatrix} = 0.$$

§ 9. *Absolute invariants.*—Two circles have a single absolute invariant:

$$C_{12} = \frac{I_{12}^2}{I_{11} I_{22}} = \cos^2 \theta_{12},$$

* Otherwise bicircular biquadratic.

† Points may be considered as degenerate circles, so that the concomitants apply also to sets of points. The condition that two points lie on a minimal line is $I_{12} = 0$; the condition that four points are cocircular is $K_{1234} = 0$; etc.

where θ_{12} is the angle of the circles. All the absolute invariants of any number of circles are functions of the quantities

$$C_{ik} = \frac{I_{ik}}{I_{ii}I_{kk}} = \cos^2\theta_{ik},$$

θ_{ik} denoting the angle of the circles f_i, f_k; but not every rational absolute invariant is a rational function of the C_{ik}'s. To obtain systems of absolute invariants possessing the latter property—which systems may be called complete *rational* systems—it is necessary to introduce the absolute invariants

$$D_{iklm} = \frac{I_{il}I_{km}}{I_{im}I_{kl}}, \quad E_{iklm} = \frac{K_{iklm}}{I_{ik}I_{lm}}. \tag{1}$$

Every rational absolute invariant of any number of circles, with respect to the mixed inversion group G', is a rational function of the quantities D_{iklm}; with respect to the continuous inversion group G, every rational absolute invariant is a rational function of the quantities E_{iklm}.

Consider now the case where the circles reduce to points. *Four points $P_1P_2P_3P_4$ have two absolute invariants*

$$\alpha = D_{1234}, \quad \beta = D_{1324}, \tag{2}$$

which completely characterize their invariant properties with respect to the group G'. With respect to the group G, the simplest absolute invariants present themselves only in the minimal system of coördinates. Let the coördinates of P_i be a_i, b_i; then the fundamental absolute invariants of the four points are

$$\sigma = \frac{(a_1 - a_3)(a_2 - a_4)}{(a_1 - a_4)(a_2 - a_3)}, \quad \tau = \frac{(b_1 - b_3)(b_2 - b_4)}{(b_1 - b_4)(b_2 - b_3)}. \tag{3}$$

The relations between (2) and (3) are

$$\alpha = \sigma\tau, \quad \beta = (1 - \sigma)(1 - \tau).$$

If the points be taken in all possible orders, six distinct pairs of values for α, β and σ, τ are found:

$$\begin{array}{lll}
1234: & \alpha, \beta; & \sigma, \tau; \\
1324: & \beta, \alpha; & 1-\sigma, 1-\tau; \\
1243: & \frac{1}{\alpha}, \frac{\beta}{\alpha}; & \frac{1}{\sigma}, \frac{1}{\tau}; \\
1342: & \frac{1}{\beta}, \frac{\alpha}{\beta}; & \frac{1}{1-\sigma}, \frac{1}{1-\tau}; \\
1423: & \frac{\beta}{\alpha}, \frac{1}{\alpha}; & \frac{\sigma-1}{\sigma}, \frac{\tau-1}{\tau}; \\
1432: & \frac{\alpha}{\beta}, \frac{1}{\beta}; & \frac{\sigma}{\sigma-1}, \frac{\tau}{\tau-1}.
\end{array} \tag{4}$$

With respect to either group G or G', the general set of four points is equivalent to itself in four ways. *The "special" sets,* i. e., *those which are equivalent to themselves in more than four ways,** are as follows:

First, as to the group G:

1°. $\sigma, \tau = 0, 0$ or $1, 1$ or ∞, ∞; equivalent to itself in 8 ways. This is the *coincident* set.

2°. $\sigma, \tau = -1, -1$ or $2, 2$ or $\frac{1}{2}, \frac{1}{2}$; equivalent to itself in 8 ways. This is the *harmonic* set.

3°. $\sigma, \tau = \omega, 1-\omega$ or $1-\omega, \omega$ where $\omega^3 = -1$; 12 ways. This is the *equianharmonic* set.

4°. $\sigma, \tau = \frac{1}{2}, \infty$ or $2, 0$ or $-1, 1$; 8 ways.

5°. $\sigma, \tau = \omega, 1-\omega$ or $1-\omega, \omega$ where $\omega^3 = -1$; 12 ways.

Second, as to the group G':

A. $\alpha, \beta = 1, \lambda$ or $\lambda, 1$ or $1/\lambda, 1/\lambda$ where $\lambda \neq 1$; 8 ways. This includes cases 1° (for which $\lambda = 0$) and 2° (for which $\lambda = 4$).

B. $\alpha, \beta = 1, 1$; 24 ways. This coincides with case 3°.

C. $\alpha, \beta = 0, -1$ or $-1, 0$ or ∞, ∞; 8 ways. This coincides with case 4°.

D. $\alpha, \beta = \epsilon, \epsilon^2$ or ϵ^2, ϵ where $\epsilon^3 = 1$; 12 ways. This coincides with case 5°.

The geometrical definitions and properties of these special sets present considerable interest, but will not be given here.

CHAPTER III.

The Binary Method.

§ 10. *Minimal coördinates.*—In the system discussed in Chapter I (tetracyclic coördinates in the plane, or tetrahedral in space), not every algebraic curve can be represented by a single equation in connection with the identity; for such an equation represents a curve cutting the minimal lines (or generators) of each system in the same number of points.† Calling the numbers which indicate how many times the lines of the two systems cut an (algebraic) curve, the partial orders of the curve, we see that the quaternary theory as developed in § 4 is restricted to the domain of "equi-ordinal" curves.‡

* It is to be noticed that this is not identical with the statement that the points admit more than four automorphic transformations: the text classification is based on the number of *equivalences*, not on the number of *transformations*. (In a classification based upon the latter, the cocircular set of four points would count as special, since such a set admits eight tranformations.)

† Conversely, every equi-ordinal curve can be represented by a single tetracyclic equation. For the general theory see Klein, *Ein liniengeometrischen Satz*, Göttingen Nachrichten, 1872; or Mathematische Annalen, vol. 22, pp. 234-241, 1882.

‡ This is not a necessary restriction of the system of tetracyclic coördinates, but only of the apolar method developed in § 4. *Cf.* § 4′, for curves whose partial orders are unequal.

This limitation is removed more conveniently than in § 4′ by the use of what may be termed the minimal system of coördinates.* In this, a point is determined by two coördinates λ, μ, the parameters of the pair of minimal lines (or generators) through the point—the parameters being ordinary projective coördinates in the minimal pencils (or hyperpencils of generators). Instead of λ, μ, we may introduce two pairs of homogeneous coördinates $\lambda_1 : \lambda_2$ and $\mu_1 : \mu_2$.

A curve whose partial orders are m, n, is represented by an equation $f(\overset{m}{\lambda}, \overset{n}{\mu}) = 0$, of degree m in λ and degree n in μ. Introducing the homogeneous coördinates, the equation of the general algebraic curve may be written

$$(1) \qquad f = f(\lambda_1, \lambda_2; \mu_1, \mu_2) = \sum_{h,k} \binom{m}{h}\binom{n}{k} a_{hk} \lambda_1^{m-h} \lambda_2^h \mu_1^{n-k} \mu_2^k = 0,$$

h and k varying from 0 to m and from 0 to n respectively. Symbolically, we may write

$$(2) \qquad f = a_\lambda^m a_\mu^n = b_\lambda^m \beta_\mu^n = \cdots,$$

where the symbols have real meaning only in the combinations:

$$a_1^{m-h} a_2^h a_1^{n-k} a_2^k = b_1^{m-h} b_2^h \beta_1^{n-k} \beta_2^k = \cdots = a_{hk}.$$

The inversion group G_6' in the present coördinates takes the form:

$$(3) \qquad \begin{aligned} G_6: &\qquad L = \frac{a\lambda + b}{c\lambda + d}, \quad M = \frac{a\mu + \beta}{\gamma\mu + \delta}; \\ H_6: &\qquad L = \frac{a\mu + b}{c\mu + d}, \quad M = \frac{a\lambda + \beta}{\gamma\lambda + \delta}. \end{aligned}$$

In the present chapter, however, attention will be restricted † to the continuous group G_6, which may also be written

$$(4)\ G_6: \quad L_1 = a\lambda_1 + b\lambda_2, \ L_2 = c\lambda_1 + d\lambda_2, \ M_1 = a\mu_1 + \beta\mu_2, \ M_2 = \gamma\mu_1 + \delta\mu_2.$$

§ 11. *Double binary forms.*—From (2) and (4) we have:

The invariant theory of curves for the group G_6, is equivalent to the theory of double binary forms whose variables undergo independent linear transformations.

The theory of double binary forms in which the variables undergo the *same* linear transformation, may be regarded as completely known; for we may pass

* The system is fixed by three base points $P'P''P'''$: denoting the corresponding minimal lines by $L'L''L'''M'M''M'''$, the coördinates of a point P are the anharmonic ratios $\lambda = (LL'L''L''')$, $u = (MM'M''M''')$. It has a theoretic advantage over the tetracyclic system, in that it introduces only invariant elements in its definition.

† See § 21 for the group G'.

from the consideration of such forms, immediately to an equivalent system of simple binary forms. In fact the concomitants of the form $r_x^m s_y^n$ (where x and y are cogredient), are the concomitants of the set: *

$$r_x^m s_x^n, \quad (rs) r_x^{m-1} s_x^{n-1}, \quad (rs)^2 r_x^{m-2} s_x^{n-2}, \quad \ldots.$$

Where, however, the variables, as in $a_\lambda^m a_\mu^n$, are not cogredient, but undergo distinct transformations, the theory is not so simple; methods, analagous to those for simple binary forms, have been developed by PEANO and GORDAN † for determining the concomitants, and it has been proved that their totality may be represented by a complete system. In the following sections a geometric basis is given to these methods, principally by the consideration of the polar theory of the forms and curves.

§ 12. *Polar theory of the general curve* $C_{m,n}$.—The double binary form

$$f = a_\lambda^m a_\mu^n,$$

equated to zero, represents a curve cutting each λ-line in n points, and each μ-line in m points—where by a λ-line is meant a minimal line for which λ is constant, etc. Such a curve will be denoted by $C_{m,n}$; so that a λ-line is a $C_{1,0}$, a μ-line a $C_{0,1}$, a circle a $C_{1,1}$, a cyclic curve a $C_{2,2}$, etc. The general $C_{m,n}$ contains $mn + m + n$ independent constants; its total order, i. e., the number of points in which it is cut by an arbitrary circle, is $m + n$. Two curves $C_{m,n}$, $C_{m',n'}$ intersect in $mn' + nm'$ points.‡ All $C_{m,n}$'s which have $mn + m + n$ points in common, will in general also have $mn - m - n$ additional points in common.

Passing over the general properties of curves, of which the above are simple types, consider now the (inversion) polars of a curve $C_{m,n}$. *The* (h, k) *polar* $P_{h,k}$ *of a point* ξ, η *with respect to a curve* $f \equiv a_\lambda^m a_\mu^n = 0$, *is defined thus:*

$$P_{h,k} \equiv a_\xi^h a_\eta^k a_\lambda^{m-h} a_\mu^{n-k} = 0; \tag{2}$$

which may also be written

$$P_{h,k} = \Delta_1^h \Delta_2^k f,$$

where

$$\Delta_1 = \frac{1}{m}\left(\xi_1 \frac{\partial}{\partial \lambda_1} + \xi_2 \frac{\partial}{\partial \lambda_2}\right), \qquad \Delta_2 = \frac{1}{n}\left(\eta_1 \frac{\partial}{\partial \mu_1} + \eta_2 \frac{\partial}{\partial \mu_2}\right). \tag{3}$$

* CLEBSCH, Göttingen Abhandlungen, vol. 17, p. 23; *Binäre Formen*, § 14.

† PEANO, Atti Accademia Torino, 1881; Giornale di Matematiche, vol. 20, pp. 79–101, 1882. GORDAN, Mathematische Annalen, vol. 23, pp. 372–389, 1889.

‡ The general $C_{m,n}$, from the standpoint of projective geometry, is the special curve of order $m + n$, (ϕ_{m+n}), which has multiple points of orders m, n at the circular points I, J, respectively; while from the point of view of inversion geometry, the general curve of order κ, (ϕ_κ), is the special $C_{\kappa,\kappa}$, which has a multiple point of order κ at infinity. Projectively, a C_{mn} and a $C_{m'n'}$ cut in $(m + n)(m' + n')$ points; but of these the mm' at I and the nn' at J are discarded in inversion geometry.

In expanded form,

$$(4) \qquad P_{h,k} = \sum_{i,j} \binom{h}{i}\binom{h}{j} \xi_1^{h-i}\xi_2^{i}\eta_1^{k-j}\eta_2^{j} f_{h-i,\,i,\,k-j,\,j}(\lambda,\ \mu) \qquad \binom{i=0,\ 1,\ \cdots m}{k=0,\ 1,\ \cdots n},$$

where

$$f_{abcd} = \frac{1}{m\cdots(m-a-b-1)n\cdots(n-c-d-1)} \cdot \frac{\partial^{a+b+c+d} f}{\partial\lambda_1^a \partial\lambda_2^b \partial\mu_1^c \partial\mu_2^d}.$$

The $(h,\ k)$ polar may also be called the polar $C_{m-h,\ n-k}$; this is put in evidence by introducing for it the double notation $P_{h,k} \equiv Q_{m-h,\ n-k}$.

The polar $C_{h,k}$ of a point $\xi,\ \eta$ is therefore:

$$Q_{h,k} = \sum_{i,j} \binom{h}{i}\binom{k}{j} f_{h-i,\,i,\,k-j,\,j}(\xi,\ \eta)\lambda_1^{h-i}\lambda_2^{i}\mu_1^{k-j}\mu_2^{j}.$$

The geometric definition of the polar curves may be obtained from the consideration of simple binary polarity in each of the minimal pencils. The curve $f = a_\lambda^m a_\mu^n$ may be regarded as establishing an $(m,\ n)$ correspondence between the system of λ-lines and the system of μ-lines: to each λ-line there corresponds the n μ-lines $\mu_1\mu_2\cdots\mu_n$ through the intersections of the λ-line with f, and to each μ-line the m λ-lines $\lambda_1\cdots\lambda_m$ through the intersections of the μ-line with f. Similarly with respect to the curve,

$$(5) \qquad a_\mu^{n-k} a_\lambda^m a_\eta^k = 0,$$

to each λ corresponds the kth polar of η as to $\mu_1\mu_2\cdots\mu_n$; there are m λ's, $\lambda_1'\lambda_2'\cdots\lambda_m'$, such that with respect to the μ's corresponding to each in f, μ belongs to the kth polar of η. Passing from (5) to

$$a_\lambda^{m-h} a_\xi^h a_\mu^{n-k} a_\eta^k = 0,$$

we obtain the required geometric definition:

The (h, k) polar of a point ξ, η with respect to a curve $C_{m,n}$ is a curve $Q_{m-h,\,n-k}$, generated by the following correspondence between the minimal pencils: to each μ correspond those λ's ($m - h$ in number) which form the h-th polar of ξ with respect to $\lambda_1'\lambda_2'\cdots\lambda_m'$—where the λ''s are defined by the fact that the μ's corresponding to each in f are such that, with respect to them, μ belongs to the k-th polar of η; and similarly to each λ, etc.

Of special importance are the polars Q_{10}, Q_{01}, Q_{11}. The polar minimal lines of a point P are

$$Q_{10} = a_\xi^{m-1} a_\eta^n a_\lambda = f_{10}\lambda_1 + f_{20}\lambda_2,$$

and

$$Q_{01} = a_\xi^m a_\eta^{n-1} a_\mu = f_{01}\mu_1 + f_{02}\mu_2 \qquad \left(f_{ik} = \frac{1}{mn}\frac{\partial^2 f}{\partial\lambda_i\partial\mu_k}\right);$$

Q_{10} is the linear polar of ξ with respect to the λ's corresponding to η in f, and Q_{01} the linear polar of η with respect to the μ's corresponding to ξ in f. To

each point P there corresponds a definite conjugate point P', the intersection of the polar minimal lines Q_{10}, Q_{01}. The number of points P which have a given point P' for conjugate is

$$mn + (m-1)(n-1) = 2mn - m - n + 1;$$

only in the case of the circle* $(m = 1, n = 1)$, therefore, is the conjugate relation involutorial.

The polar circle of P is,

$$Q_{11} = a_\xi^{m-1} a_\eta^{n-1} a_\lambda a_\mu = f_{11}\lambda_1\mu_1 + f_{12}\lambda_1\mu_2 + f_{21}\lambda_2\mu_1 + f_{22}\lambda_2\mu_2.$$

It is generated by a $(1, 1)$ correspondence: to each μ corresponds the linear polar of ξ with respect to $\lambda_1' \cdots \lambda_m'$, where the λ''s are such that with respect to the μ's corresponding in f, μ is the linear polar of η.

Between the polar minimal lines, the polar circles, and the conjugate point of any point P with respect to f, there exists this relation: the pair of lines through the given point is cut in the same two points by the polar circle and by the polar minimal lines; or the conjugate of P is the inverse of P with respect to the polar circle.

The general properties of the polar curves may be developed by practically the same methods as in the projective theory. If the point B lies on the (h, k) polar of the point A, A lies on the $(m-h, n-k)$ polar of B. The polar C_{hk} of P with respect to f is also the polar C_{hk} of P with respect to any polar of f. The (hk) polar of P with respect to the $(h'k')$ polar of P is the $(h+h', k+k')$ polar of P. If P lies on f, all its polars are tangent to f at P, etc.

§ 13. *Transvectants.*—In the theory of simple binary forms, the most important process for the formation of invariants is that of transvection (Ueberschiebung): CLEBSCH and GORDAN have shown that this single process, by repeated application, will yield the totality of concomitants. An analogous process of double transvection (doppio scorrimento), has been applied by PEANO† to the double binary forms.

Let any two forms (not necessarily distinct) be

$$f = a_\lambda^m a_\mu^n, \quad \phi = b_\lambda^{m'} \beta_\mu^{n'};$$

their (hk) *double transvectant* is defined by

$$(f\phi)_{hk} = (ab)^h (a\beta)^k a_\lambda^{m-h} b_\lambda^{m'-h} a_\mu^{n-k} \beta_\mu^{n'-k}. \tag{1}$$

* In this case the relation of conjugacy coincides with inversion.

† PEANO, *Formazioni invariantive della correspondenza*, Giornale di Matematiche, vol. 20, p. 79, 1882.

The non-symbolic expression is

$$(f\phi)_{hk} = [\Omega_1^h \Omega_2^k f(\lambda\mu)\phi(\xi\eta)]_{\xi,\,\eta=\lambda,\,\mu} \tag{2}$$

where Ω_1, Ω_2 are the operators defined by *

$$\Omega_1 = \frac{1}{mm'}\left(\frac{\partial^2}{\partial\lambda_1\partial\xi_2} - \frac{\partial^2}{\partial\lambda_2\partial\xi_1}\right), \quad \Omega_2 = \frac{1}{nn'}\left(\frac{\partial^2}{\partial\mu_1\partial\eta_2} - \frac{\partial^2}{\partial\mu_2\partial\eta_1}\right).$$

Transvectants are covariants which are linear in the coefficients of each form. If f and ϕ have the same orders m, n, the (mn) transvectant is an invariant

$$(f\phi)_{mn} = (ab)^m(a\beta)^n,$$

which, as in the case of simple binary forms, may be called the bilinear, harmonic or apolar invariant of the forms.

THEOREM. *The locus of points whose polar $C_{h,k}$'s with respect to f and ϕ are apolar is $(f\phi)_{hk} = 0$.*

For the polar $C_{h,k}$'s are

$$Q^f_{hk} = a_\xi^{m-h} a_\eta^{n-k} a_\lambda^h a_\mu^k,$$

$$Q^\phi_{hk} = b_\xi^{m'-h} \beta_\eta^{n'-k} b_\lambda^h \beta_\mu^k;$$

therefore their apolar invariant is

$$(Q^f_{hk},\ Q^\phi_{hk})_{hk} = (ab)^h(a\beta)^k a_\xi^{m-h} b_\xi^{m'-h} a_\eta^{n-k} \beta_\eta^{n'-k}.$$

When f and ϕ coincide, the theorem becomes: the locus of points whose polar C_{hk}'s with respect to f are self-apolar is $(ff)_{hk} = 0$.

Consider now the simplest transvectants, which play the rôle here of Jacobians and Hessians in projective geometry:

$$(f\phi)_{10} = (ab)a_\lambda^{m-1} b_\lambda^{m'-1} a_\mu^n \beta_\mu^{n'} = f_{10}\phi_{20} - f_{20}\phi_{10},$$

$$(f\phi)_{01} = (a\beta)a_\lambda^m b_\lambda^{m'} a_\mu^{n-1} \beta_\mu^{n'-1} = f_{01}\phi_{02} - f_{02}\phi_{01}.$$

The curve $(f\phi)_{10} = 0$ is the locus of points whose polar λ-lines with respect to f and ϕ coincide; for the points of $(f\phi)_{01} = 0$ the polar μ lines coincide.

If a point lies on both these transvectants, it has the same minimal lines with respect to each of the curves, therefore:

There are $2(m+m')(n+n') - 2(m+n+m'+n') + 4$ points whose conjugate points with respect to f and ϕ coincide; these are the intersections of $(f\phi)_{10}$ and $(f\phi)_{01}$, and include the $mn' + m'n$ intersections of f and ϕ.

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* $$\Omega_1^h = \frac{1}{m(m-1)\cdots(m-h+1)m'(m'-1)\cdots(m'-h+1)}\left(\frac{\partial^2}{\partial\lambda_1\partial\xi_2} - \frac{\partial^2}{\partial\lambda_2\partial\xi_1}\right)^h.$$

$$\Omega_2^k = \frac{1}{n(n-1)\cdots(n-k+1)n'(n'-1)\cdots(n'-k+1)}\left(\frac{\partial^2}{\partial\mu_1\partial\eta_2} - \frac{\partial^2}{\partial\mu_2\partial\eta_1}\right)^k.$$

The (1, 1) transvectant is

$$(f\phi)_{11} = (ab)(a\beta)a_\lambda^{m'-1}b_\lambda^{m'-1}a_\mu^{n-1}\beta_\mu^{n'-1}$$
$$= f_{11}\phi_{22} - f_{12}\phi_{22} - f_{21}\phi_{12} + f_{22}\phi_{11};$$

or, when the forms coincide,

$$(ff)_{11} = (aa')(aa')a_\lambda^{m-1}a_\lambda'^{m-1}a_\mu^{n-1}a_\mu'^{n-1}$$
$$= 2f_{11}f_{22} - 2f_{12}f_{21}.$$

For this case the first theorem of this section gives:

The (1, 1) *transvectant of two curves is the locus of points whose polar circles with respect to the two curves are orthogonal.*

The (1, 1) *transvectant of a curve over itself is the locus of points whose polar circles are degenerate. Therefore* $(ff)_{11} = 0$ *cuts* f *in its minimal points, the number of the latter being* $4mn - 2m - 2n$. By a minimal point of a curve is meant one where the tangent line is a minimal line. At such a point the polar circle consists of the pair of minimal lines through the point.

§14. *Some simultaneous covariants.*—Denote the polar circles of a point P with respect to the curves f, ϕ, $\psi \cdots$ by C_f, C_ϕ, $C_\psi \cdots$; so that

$$C_f = a_\xi^{m-1}a_\eta^{n-1}a_\lambda a_\mu = f_{11}\lambda_1\mu_1 + f_{12}\lambda_1\mu_2 + f_{21}\lambda_2\mu_1 + f_{22}\lambda_2\mu_2, \text{ etc.}$$

Then, from § 6 and § 13,

(1)
$$I(C_f, C_f) = (ff)_{11}, \qquad I(C_f C_\phi) = (f\phi)_{11},$$
$$K(C_f C_\phi C_\psi C_\chi) = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ \phi_{11} & \phi_{12} & \phi_{21} & \phi_{22} \\ \psi_{11} & \psi_{12} & \psi_{21} & \psi_{22} \\ \chi_{11} & \chi_{12} & \chi_{21} & \chi_{22} \end{vmatrix} = S_1.$$

Any point P may be considered as a degenerate circle, i. e., as the circle with the equation $\overline{P} = (\xi\lambda)(\eta\mu) = 0$; adding this to the system of polar circles, we have

$$I(C_f\overline{P}) = a_\xi a_\eta a_\xi^{m-1}a_\eta^{n-1} = f,$$
$$K(C_f C_\phi C_\psi \overline{P}) = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ \phi_{11} & \phi_{12} & \phi_{21} & \phi_{22} \\ \psi_{11} & \psi_{12} & \psi_{21} & \psi_{22} \\ \lambda_2\mu_2 & -\lambda_2\mu_1 & -\lambda_1\mu_2 & \lambda_1\mu_1 \end{vmatrix} = S_2.$$

From these formulæ we can write down at once any covariant which is defined by a property of the polar circles and $\overline{P}$. Thus the locus of points such that the polar circles with respect to f and ϕ are tangent, is $(ff)_{11}(\phi_1\phi)_{11} - (f\phi)^2_{11} = 0$ (§ 8). The locus defined by C_f, C_ϕ and $\overline{P}$ having a common point is

$$\begin{vmatrix} (ff)_{11} & (f\phi)_{11} & f \\ (\phi f)_{11} & (\phi\phi)_{11} & \phi \\ f & \phi & 0 \end{vmatrix} = 0 .$$

The defining property of S_1 above is that $C_f C_\phi C_\psi C_\chi$ have a common orthogonal circle. Symbolically,

$$S_1 = \{(ac)(bd)(a\beta)(\gamma\delta) - (ab)(cd)(a\gamma)(\beta\delta)\} a_\lambda^{m-1} a_\mu^{n-1} \cdots d_\lambda^{m'''-1} \delta_\mu^{n'''-1} .$$

Its square is expressible in simple transvectants thus:

$$S_1^2 = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \times \begin{vmatrix} f_{22} & f_{21} & f_{12} & f_{11} \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} (ff)_{11} & (f\phi)_{11} & (f\psi)_{11} & (f\chi)_{11} \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} .$$

Similarly S_2 is the locus of points $\overline{P}$ for which $C_f C_\phi C_\chi$ and $\overline{P}$ have a common orthogonal circle;

$$S_2 = \{(ac)(a\beta)b_\lambda\gamma_\mu - (ab)(a\gamma)c_\lambda\beta_\mu\} a_\lambda^{m-1} a_\mu^{n-1} b_\lambda^{m'-1} \beta_\mu^{n'-1} c_\lambda^{m''-1} \gamma_\mu^{n''-1} ;$$

and

$$(2) \qquad S_2^2 = \begin{vmatrix} (ff)_{11} & (f\phi)_{11} & (f\psi)_{11} & f \\ \cdot & \cdot & \cdot & \cdot \\ f & \phi & \psi & 0 \end{vmatrix} .$$

§ 15. *The discriminants and the nodal invariant.*—If in $f = a_\lambda^m a_\mu^n$ we consider λ or μ as being the only variable, and take the corresponding discriminants, we obtain two simple binary forms $D_1(\lambda)$, $D_2(\mu)$, which may be called the *discriminants* * of f. Let the symbolic discriminant of a_μ^n be $\Delta_1(a, \beta, \cdots)$; it will contain in all $2(n-1)$ symbols $a, \beta, \cdots$. Similarly the discriminant $\Delta_2(a, b, \cdots)$ of a_λ^m contains $2(m-1)$ symbols; D_1 and D_2 are then

$$(1) \qquad \begin{aligned} D_1(\lambda) &= \Delta_1(a, \beta, \cdots) a_\lambda^m b_\lambda^m \cdots , \\ D_2(\mu) &= \Delta_2(a, b, \cdots) a_\mu^n \beta_\mu^n \cdots , \end{aligned}$$

their orders being $2(n-1)m$ and $2(m-1)n$. They represent the tangent minimal lines of f, the total number of such lines being $4mn - 2m - 2n$ as found before (§ 13).

* CAPELLI, *Sopra la corrispondenza* (2, 2), Giornale di Matematiche, vol. 17, p. 75, 1879.

We consider now certain invariants of f which have simple geometric interpretations. Let $\delta_1 = 0$ be the condition for two λ-tangents following together; $P_1 = 0$ the condition for an inflexional λ-tangent; $Q_1 = 0$ the condition for a double λ-tangent; and $\delta_2 = 0$, $P_2 = 0$, $Q_2 = 0$ the corresponding conditions for μ-tangents.

From geometric considerations,

$$\delta_1 = N^a P_1^b Q_1^c, \quad \delta_2 = N^a P_2^b Q_2^c.$$

To determine a, b, c we must calculate the degrees of the invariants. The invariant δ_1 is the discriminant of $D_1(\lambda)$; its degree is therefore

$$4(n-1)\{2m(n-1)-1\}.$$

Again P_1, Q_1, equated to 0, are the conditions for the existence of a λ for which the equation $f(\lambda, \mu) = 0$ has a triple root, and two double roots respectively; their degrees may be found then, from SALMON's general formulæ for the order and weight of systems of equations.* It is thus found that the degree of P_1 is $6m(n-2)$, and the degree of Q_1 is $4m(n-2)(n-3)$.

To determine a, b, c, we have then the equations:

$$4(n-1)\{2m(n-1)-1\} = a[N] + 6bm(n-2) + 4cm(n-2)(n-3),$$

$$4(m-1)\{2n(m-1)-1\} = a[N] + 6bn(m-2) + 4cn(m-2)(m-3),$$

where $[N]$ denotes the degree of N. Equating the values of $a[N]$, we obtain $b = 3$, $c = 2$; and determining a from the values of the invariants for the circle, we have

$$\delta_1 = NP_1^3 Q_1^2, \quad \delta_2 = NP_2^3 Q_2^2. \tag{2}$$

The nodal invariant is the greatest common divisor of the discriminants δ_1, δ_2, of the discriminants D_1, D_2, of f. Its degree is $6mn - 4(m+n-1)$. †

§ 16. *Contravariants.*—Besides invariants and covariants, forms involving the coördinates of a variable circle are also of geometric importance; such a form, equated to zero, denotes a complex of circles with invariant relations to the given curve or curves. The variable circle will be taken in the form:

$$C = r_{11}\lambda_1\mu_1 + r_{12}\lambda_1\mu_2 + r_{21}\lambda_2\mu_1 + r_{22}\lambda_2\mu_2 = r_\lambda \rho_\mu. \tag{1}$$

*SALMOND-FIELDER, *Raumgeometrie*, 3d edition, vol. II, p. 595, formulæ for cases *c*) and *d*).

† The intersection of two surfaces can have a node only if the surfaces are tangential. Therefore in the case where f is equi-ordinal, $f = a_\lambda^n a_\mu^n$, the problem of finding N is equivalent to the problem of finding the tact-invariant of a quadric and the general surface of order n. The above result then gives a method of finding this tact-invariant by purely binary considerations. The degree is $6n^2 - 8n + 4$, as may also be found from quaternary considerations (SALMON-FIELDER, *Raumgeometrie*, 3d edition, II, 609).

The circle C intersects $f = a_\lambda^m a_\mu^n$ in $m + n$ points, whose coördinates are obtained by the elimination of λ or μ from $C = 0$, $f = 0$; thus we obtain two binary forms:

$$(2)\qquad \begin{aligned} L(\lambda) &= (f,\ C^n)_{0n} = (a\rho')\cdots(a\rho^{(n)})a_\lambda^m r_\lambda'\cdots r_\lambda^{(n)} = l_\lambda^{m+n}, \\ M(\mu) &= (f,\ C^m)_{m0} = (ar')\cdots(ar^{(m)})a_\mu^n \rho_\mu'\cdots\rho_\mu^{(m)} = l_\mu^{m+n}, \end{aligned}$$

the first representing the λ-lines, and the second the μ-lines, through the intersections of C and f. That it is necessary in general to consider only one of these forms is shown by the theorem:

If C is not degenerate, the forms L and M are equivalent.

This is proved by showing that the substitution

$$(3)\qquad \begin{aligned} \lambda_1 &= \quad r_{21}\mu_1 + r_{22}\mu_2 = \quad r_2\rho_\mu, \\ \lambda_2 &= -r_{11}\mu_1 - r_{12}\mu_2 = -r_1\rho_\mu, \end{aligned}$$

whose modulus

$$-\tfrac{1}{2}(rr')(\rho\rho') = -\tfrac{1}{2}\sigma'$$

is not zero by assumption—transforms L into M except for a factor. We have

$$L' = (a\rho')\cdots(a\rho^{(n)})\bar{\rho}_\mu'\cdots(a\bar{r}^{(m)})\bar{\rho}_\mu^{(m)}(\bar{r}'\bar{r}')\bar{\rho}_\mu'\cdots(\bar{r}^{(m)}\bar{r}^{(n)})\bar{\rho}_\mu^{(n)};$$

but

$$(\bar{r}\bar{r})(a\rho)\bar{\rho}_\mu = -\sigma' a_\mu;$$

therefore

$$L' = (-1)^n\sigma'^n M.$$

The equivalence may also be seen geometrically; for L and M on the circle C represent the same points in different systems of linear coördinates.

From L or M a large class of contravariants may be deduced, since every invariant of L or M is a contravariant of f. Thus *to every invariant of a binary form of order $m + n$, corresponds a contravariant of the double binary form $f = a_\lambda^m a_\mu^n$*; this contravariant represents the complex of circles, each of which cuts f in $m + n$ points having the property denoted by the vanishing of the binary invariant.*

As an application consider the discriminants D_L and D_M, of L and M. The discriminant D_L can vanish only if the circle C is tangent to f, or if two of the points of intersection lie on a λ-line—the latter case occurring only where C is degenerate. If we denote the complex of tangent circles by Φ, we have

$$D_L = \Phi^n\sigma'^b.$$

* Compare the *Clebsch Uebertragungsprincip*, LINDEMANN, *Geometrie*, vol. I, p. 276. Similarly every covariant of the binary form gives a mixed concomitant of f, i. e., a concomitant involving the coördinates of both a variable point and a variable circle.

Reasoning as in the preceding article, we find

$$D_L = \Phi\sigma'^{n(n-1)}, \quad D_M = \Phi\sigma'^{m(m-1)}. \tag{4}$$

The tangential complex Φ may therefore be obtained by forming either discriminant D_L or D_M, and neglecting the irrelevant complex of degenerate circles. Its degree (in coefficients of f) is $2(m+n-1)$, and its class (degree in coefficients of C) is $2mn$.*

Another method which is useful in the formation of contravariants, is founded upon a principle by which from any equi-covariant, i. e., a covariant which is equi-ordinal—we may obtain a corresponding contravariant; and conversely, from any contravariant, a corresponding covariant. Any contravariant Ω of a system of forms can be written

$$\Omega = (tr)(tr')\cdots(tr^{(k-1)})(\tau\rho)(\tau\rho')\cdots(\tau\rho^{(k-1)}). \tag{5}$$

If in (5) we put $(tr) = t_\lambda$, $(\tau\rho) = \tau_\mu$, we obtain a covariant

$$V = t_\lambda^k \tau_\mu^k, \tag{6}$$

involving λ and μ in the same orders; and similarly, from any equi-covariant (6) a corresponding contravariant (5) may be derived.

From any contravariant Ω, a covariant V may be obtained by substituting point coördinates λ, μ for the symbols r, ρ of the variable circle which occurs in Ω; and from any equi-covariant V, a contravariant Ω may be obtained by substituting for λ, μ the symbols of a circle.

Using the definition of the polar circle of a given circle with respect to a curve, which is given later (§ 24), the geometric relations of corresponding covariants and contravariants are:

$V = 0$ is the locus of the vertices of the degenerate circles contained in $\Omega = 0$; while $\Omega = 0$ is the complex of circles which are orthogonal to their polar circles with respect to the curve $V = 0$.

CHAPTER IV.

Connection of the Binary and Quaternary Theories.

In the preceding, the binary and quaternary methods have been considered independently of each other; but the fact that both are methods which apply to

* The class of the tangential complex may also be obtained from projective considerations. The class of Φ is the number of circles in any pencil of circles, which are tangent to f; it is therefore also the number of lines through a point, which are tangent to f, i. e., it is the (projective) class of the curve f, which from Plucker's formula is

$$(m+n)(m+n-1) - m(m-1) - n(n-1) = 2mn.$$

the same geometry, indicates a certain isomorphism between the two. The explicit form of this connection will be studied in the present section. It will be shown how the apolar surface which proved to be fundamental in the quaternary theory, also presents itself from the binary point of view; and how we may pass from a quaternary concomitant to a corresponding binary concomitant.

§ 17. *Parametric representation of the fundamental quadric.*—Consider any four circles not belonging to a linear system,

$$C_1 = \overline{a}_\lambda \overline{a}_\mu, \quad C_2 = \overline{b}_\lambda \overline{\beta}_\mu, \quad C_3 = \overline{c}_\lambda \overline{\gamma}_\mu, \quad C_4 = \overline{d}_\lambda \overline{\delta}_\mu. \tag{1}$$

In connection with these there are the covariant orthogonal circles

$$C_1' = \overline{\overline{a}_\lambda \overline{a}_\mu}, \quad C_2' = \overline{\overline{b}_\lambda \overline{\beta}_\mu}, \quad C_3' = \overline{\overline{c}_\lambda \overline{\gamma}_\mu}, \quad C_4' = \overline{\overline{d}_\lambda \overline{\delta}_\mu}, \tag{2}$$

where for example C_1' is the circle orthogonal to $C_2 C_3 C_4$, so that

$$\overline{\overline{a}_\lambda \overline{a}_\mu} = (\overline{b}\,\overline{d})(\overline{\beta}\,\overline{\gamma})\overline{c}_\lambda \overline{\delta}_\mu - (\overline{b}\,\overline{c})(\overline{\beta}\,\overline{\delta})\overline{d}_\lambda \overline{\gamma}_\mu, \quad \text{etc.;}$$

and the invariants

$$I_{11} = (\overline{a}\,\overline{a}'), \quad I_{12} = (\overline{a}\,\overline{b})(\overline{a}\,\overline{\beta}), \quad \text{etc.,} \tag{3}$$

$$I_{11}' = (\overline{\overline{a}\,\overline{a}'}), \quad I_{12}' = (\overline{\overline{a}\,\overline{b}})(\overline{\overline{a}\,\overline{\beta}}), \quad \text{etc.,}$$

or

$$I_{ik}' = \text{minor of } I_{ik} \text{ in the determinant } |I_{ik}|,$$

$$D = K_{1234} = (\overline{a}\,\overline{c})(\overline{b}\,\overline{d})(\overline{a}\,\overline{\beta})(\overline{\gamma}\,\overline{\delta}) - (\overline{a}\,\overline{b})(\overline{c}\,\overline{d})(\overline{a}\,\overline{\gamma})(\overline{\beta}\,\overline{\delta}), \quad D^2 = |I_{ik}'|, \tag{4}$$

$$D' = K_{1234}' = D^3, \quad D'^2 = |I_{ik}'| = D^6.$$

Let $x_1 x_2 x_3 x_4$ be homogeneous coördinates in space, and put

$$\rho x_1 = \overline{a}_\lambda \overline{a}_\mu, \quad \rho x_2 = \overline{b}_\lambda \overline{\beta}_\mu, \quad \rho x_3 = \overline{c}_\lambda \overline{\gamma}_\mu, \quad \rho x_4 = \overline{d}_\lambda \overline{\delta}_\mu. \tag{5}$$

This is the parametric representation of a quadric; for the four circles $C_1 C_2 C_3 C_4$ are connected by the identity (§7):

$$\begin{vmatrix} |I_{ik}|, & C_i \\ C_k, & 0 \end{vmatrix} \equiv 0,$$

where the left hand member denotes the fifth order determinant obtained by bordering the fourth order determinant $|I_{ik}|$.

The equation of the quadric (5) is therefore

$$Q = p_x^2 = -\begin{vmatrix} |I_{ik}|, & x_i \\ x_k, & 0 \end{vmatrix} = \sum I_{ik}' x_i x_k = 0. \tag{6}$$

This represents the most general non-degenerate quadric; for the invariants I'_{ik} are independent, and the discriminant

$$\Delta = |I'_{ik}| = D^6 = \tfrac{1}{24}(pp'p''p''')^2, \tag{7}$$

by the assumption that $C_1C_2C_3C_4$ are linearly independent, can not vanish. Again the x's may be regarded as the tetracyclic coördinates of the point λ, μ, the base circles being $C_1C_2C_3C_4$, and the identity being $Q \equiv 0$.

The tangential equation of Q is,

$$\bar{Q} = u_P^2 = \tfrac{1}{6}(pp'p''u)^2 = D^4 \sum I_{ik} u_i u_k = 0. \tag{8}$$

Any circle $C = r_\lambda \rho_\mu$ is connected with $C_1C_2C_3C_4$ by the identity [(4) § 7],

$$Dr_\lambda \rho_\mu \equiv (r\bar{a})(\rho\bar{a})C_1 + (r\bar{b})(\rho\bar{\beta})C_2 + \text{etc.}; \tag{9}$$

therefore its quaternary coördinates are

$$u_1 = (\bar{a}r)(\bar{a}\rho), \quad u_2 = (\bar{b}r)(\bar{\beta}\rho), \quad u_3 = (\bar{c}r)(\bar{\gamma}\rho), \quad u_4 = (\bar{d}r)(\bar{\delta}\rho). \tag{10}$$

The formulæ (5) institute a (1, 1) correspondence between the points (λ, μ) of a plane and points x_i of the quadric (6). In this correspondence the points of a circle in the plane correspond to the points of a plane section of Q; the formulæ (10) show how the coördinates of corresponding circles and planes are connected.

§ 18. *The apolar surface corresponding to an equi-form.*—Consider any double binary form

$$f = a_\lambda^n a_\mu^n \tag{1}$$

whose partial orders are equal. Such an "equi-form" may be expressed in terms of the four bilinear forms $C_1C_2C_3C_4$. For by (9) § 17,

$$Da_\lambda a_\mu \equiv (a\bar{a})(a\bar{a})\bar{a}_\lambda \bar{a}_\mu + \text{etc.};$$

therefore,

$$\begin{aligned} D^n a_\lambda^n a_\mu^n \equiv {} & [(a\bar{a}')(a\bar{a}')\bar{a}'_\lambda \bar{a}'_\mu + (a\bar{b}')(a\bar{\beta}')\bar{b}'_\lambda \bar{\beta}'_\mu + \cdots] \\ & \times [(a\bar{a}'')(a\bar{a}'')\bar{a}''_\lambda \bar{a}''_\mu + \cdots \quad \cdots \quad \cdots \quad \cdots] \\ & \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ & \times [(a\bar{a}^{(n)})(a\bar{a}^{(n)})\bar{a}^{(n)}_\lambda \bar{a}^{(n)}_\mu + \cdots \quad \cdots \quad \cdots]. \end{aligned} \tag{2}$$

Introducing the x's by (1) § 17, we have

$$\left\{ \begin{array}{l} D^n a_\lambda^n a_\mu^n = A_x^n, \\ \text{where the symbols } A \text{ are defined by} \\ A_1 = (a\bar{a})(a\bar{a}) = (a\bar{a}')(a\bar{a}') = \cdots, \quad A_2 = (\bar{a}b)(\bar{a}\beta) = (\bar{a}b')(\bar{a}\beta') = \cdots, \\ A_3 = (a\bar{c})(a\bar{\gamma}) = (a\bar{c}')(a\bar{\gamma}') = \cdots, \quad A_4 = (a\bar{d})(a\bar{\delta}) = (a\bar{d}')(a\bar{\delta}') = \cdots. \end{array} \right. \tag{3}$$

Starting with the binary form $f = a_\lambda^n a_\mu^n$, we obtain in this way a perfectly definite quaternary form $F = A_x^n$. For all points of the quadric Q, the form F vanishes when and only when f does—that is to say, in space, $F = 0$ represents a surface which cuts the quadric Q in the curve $f = 0$,—but obviously this alone does not determine F. The determination is completed in the

FUNDAMENTAL THEOREM.—*The quaternary form $F = A_x^n$, which corresponds* [by (3)] *to a double binary form $f = a_\lambda^n a_\mu^n$, represents that surface through the curve $f = 0$ on the quadric Q, which is apolar to Q.*

To prove this, consider the covariant $A_P^2 A_x^{n-2}$:

$$A_P^2 = D^4 \sum I_{ik} A_i A_k = - \begin{vmatrix} |I'_{ik}| & A_i \\ A_k & 0 \end{vmatrix}$$

$$= \begin{vmatrix} |I'_{ik}| & (a\bar{a})(a\bar{a}) \\ (a\bar{a}')(a\bar{a}') & (aa)(aa) \end{vmatrix} + (aa)(aa)\,|I'_{ik}| = 0,$$

by § 3; therefore $A_P^2 A_x^{n-2} \equiv 0$, and F is apolar to Q (§ 3).

This result gives a method for determining the apolar surface F passing through a curve on Q, which is essentially different from that in § 9. Let the curve be determined by any surface $F' = 0$ which passes through it; by the substitution (5) § 17, F' becomes a double binary form $f = a_\lambda^n a_\mu^n$; then by (3) we pass from f to a quaternary form F which will be the required apolar form.

If Q is taken in the form

$$Q = 2x_1x_4 - 2x_2x_3,$$

the parametric representation may be put into the normal form

$$x_1 = \lambda_1\mu_1, \quad x_2 = \lambda_1\mu_2, \quad x_3 = \lambda_2\mu_1, \quad x_4 = \lambda_2\mu_2.$$

Then the F corresponding to $f = a_\lambda^n a_\mu^n$ is

$$F = (a_1a_1x_1 + a_1a_2x_2 + a_2a_1x_3 + a_2a_2x_4)^n;$$

and the passage from F' to F may be formulated as follows: let

$$F' = \sum c C x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta,$$

where c denotes the numerical and C the literal coefficients; substitute $x_1 = \lambda\mu$, $x_2 = \lambda$, $x_3 = \mu$, $x_4 = 1$, so that F' becomes, say,

$$\sum (c_1C_1 + c_2C_2 + \cdots)\lambda^r\mu^s;$$

in this put $C_1 = C_2 =$ etc.; the result will be F.

§ 19. *Coincidence of the binary and quaternary polar theories.*—It has been seen in chapter II that the study of a curve in tetracyclic coördinates depends

intimately upon the apolar form F; and by means of the quaternary polars of F, a complete quaternary polar theory of curves might have been developed. This was not done however, only binary polars being considered in § 12. The question may then arise, whether curves have two distinct inversion polar theories, a binary and a quaternary.

The apolar form corresponding to

$$f = a_\lambda^n a_\mu^n,$$

is, by (3),

$$F = A_x^n = [(a\bar{a}')(a\bar{a}')x_1 + \cdots][(a\bar{a}'')(a\bar{a}'')x_1 + \cdots]\cdots[(a\bar{a}^{(n)})(a\bar{a}^{(n)})x_1 + \cdots].$$

For the points of Q, we have

$$A_x^n = D^n a_\lambda^n a_\mu^n, \quad A_y = D a_\xi^n a_\eta,$$

$$A_y^k A_x^{n-k} = D^n a_\xi^k a_\eta^k a_\lambda^{n-k} a_\mu^{n-k},$$

where y or ξ, η is any point of Q. Again, since F is apolar to Q, so are all its polars $A_y^k A_x^{n-k}$. Therefore

If f and F are corresponding forms, so are their polars $P_{kk}(f)$ and $P_k(F)$.

This gives a characteristic property of the apolar surface F. *The polar surfaces of any point of Q with respect to the surface F intersect Q in the (equi-ordinal) polars of the same point with respect to the curve f;* or briefly, *the two polar theories coincide.**

§20. *Symbolic relations.*†—The object of this section is to develop the formulæ by which, from a quaternary concomitant, we may pass to a corresponding binary concomitant and conversely. The fundamental quadric, its contravariant and its discriminant are

$$(1) \quad Q = p_x^2, \quad \overline{Q} = u_P^2 = \tfrac{1}{6}(pp'p''u)^2, \quad \Delta = \tfrac{1}{24}(pp'p''p''')^2 = \tfrac{1}{4}p_P^2 = D^6.$$

The symbols of binary forms, corresponding to quaternary forms whose symbols are $A, B, C\cdots$, will be written a, a; b, β; c, γ; $\cdots$, so that by (3) § 18,

$$(2) \quad \begin{aligned} A_1 &= (a\bar{a})(a\bar{a}), \quad A_2 = (a\bar{b})(a\bar{\beta}), \quad \cdots, \\ B_1 &= (b\bar{a})(\beta\bar{a}), \quad B_2 = (b\bar{b})(\beta\bar{\beta}), \quad \cdots, \\ C_1 &= (c\bar{a})(\gamma\bar{a}), \quad C_2 = (c\bar{b})(\gamma\bar{\beta}), \quad \cdots, \quad \text{etc.} \end{aligned}$$

* Attention is restricted here to the equi-ordinal binary polars, since the others can not be represented by a single quaternary form; if we include all the polars, the quaternary theory forms merely a part of the binary theory.

† For relations connecting ternary and simple binary forms, Cf. LINDEMANN, *Sur une représentation géométrique des covariants des formes binaires*, Bulletin Société Mathématique de France, vol. 5, pp. 113-125, 1876, vol. 6, pp. 195-208, 1877; also Mathematische Annalen, vol. 23.

From these we have the fundamental relations:

$$(3)\qquad (ABCD) = \begin{vmatrix} (a\bar{a})(a\bar{a}), & (a\bar{b})(a\bar{\beta}), & \cdots \\ (b\bar{a})(\beta\bar{a}), & (b\bar{b})(\beta\bar{\beta}), & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \begin{vmatrix} a_1a_1 & a_1a_2 & a_2a_1 & a_2a_2 \\ b_1\beta_1 & b_1\beta_2 & b_2\beta_1 & b_2\beta_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \times \begin{vmatrix} \bar{a}_1\bar{a}_1, & \bar{a}_1\bar{a}_2, & \bar{a}_2\bar{a}_1, & \bar{a}_2\bar{a}_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= D^3\{(ac)(bd)(a\beta)(\gamma\delta) - (ab)(cd)(a\gamma)(\beta\delta)\} \qquad [(4)\ \S\ 17];$$

$$(4)\qquad A_PB_P = \tfrac{1}{6}(pp'p''A)(pp'p''B) = D^4 \sum I_{ik}A_iB_k$$

$$= -\begin{vmatrix} |I'_{ik}| & A_i \\ B_k & 0 \end{vmatrix} = -\begin{vmatrix} |I'_{ik}| & (a\bar{a})(a\bar{a}) \\ (b\bar{a})(\beta\bar{a}) & (ab)(a\beta) \end{vmatrix} + (ab)(a\beta)|I'_{ik}|$$

$$= D^6(ab)(a\beta) \qquad [(5)\ \S\ 7 \text{ and } (4)\ \S\ 17];$$

$$(5)\qquad \begin{cases} (pp'AB)(pp'CD) = 2D^6 \begin{vmatrix} (ac)(a\gamma) & (ad)(a\delta) \\ (bc)(\beta\gamma) & (bd)(\beta\delta) \end{vmatrix}, \\ (pABC)(p'DEF) = -D^6 \begin{vmatrix} (ad)(a\delta), & (ae)(a\epsilon), & (af)(a\phi) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}, \\ (ABCD)(A'B'C'D') = D^6 \begin{vmatrix} (aa')(aa'), & (ab')(a\beta'), & (ac')(a\gamma'), & (ad')(a\delta') \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}, \\ = D^6\{(ac)(bd)(a\beta)(\gamma\delta) - (ab)(cd)(a\gamma)(\beta\delta)\}\{(a'c')(b'd')(a'\beta')(\gamma'\delta') \\ \qquad - (a'b')(c'd')(a'\gamma')(\beta'\delta')\}; \end{cases}$$

$$(6)\qquad A_x = Da_\lambda a_\mu;$$

$$(pABC)p_x = \tfrac{1}{2}\begin{vmatrix} \dfrac{\partial Q}{\partial x_1} & \dfrac{\partial Q}{\partial x_2} & \dfrac{\partial Q}{\partial x_3} & \dfrac{\partial Q}{\partial x_4} \\ A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \end{vmatrix};$$

and on substituting

$$\tfrac{1}{2}\frac{\partial Q}{\partial x_1} = I'_{11}x_1 + \cdots + I'_{14}x_4 = \begin{vmatrix} \overline{a}_\lambda \overline{a}_\mu & \overline{b}_\lambda \overline{\beta}_\mu & \cdots \\ I_{21} & I_{22} & \cdots \\ I_{31} & I_{32} & \cdots \\ I_{41} & I_{42} & \cdots \end{vmatrix} = DC'_1, \text{ etc.} \quad [(2)\,\S\,17],$$

we obtain

$$(7) \quad (pABC)p_x = D \begin{vmatrix} C'_1 & C'_2 & \cdots \\ A_1 & A_2 & \cdots \\ B_1 & B_2 & \cdots \\ C_1 & C_2 & \cdots \end{vmatrix} = D^4\{(ac)(a\beta)b_\lambda\gamma_\mu - (ab)(a\gamma)c_\lambda\gamma_\mu\}.$$

§ 21. *Transformation of the concomitants.*—The general G concomitant of a system of quaternary forms can be written (§ 1)

$$S = \sum\phi(p)T,$$

where T represents the general projective concomitant of the quaternary forms and the fundamental quadric. We may therefore apply the Clebsch-Aronhold symbolism: every G concomitant of the form A_x^n is symbolically a concomitant of the linear forms $A_x, B_x, \cdots$. From § 5, then, S can be symbolically expressed in terms of the types

$$(1) \qquad \begin{aligned} O&: \quad A_x,\ A_PB_P,\ A_Pu_P; \\ O'&: \quad (ABCD),\ (ABCp)p_x,\ (ABCu). \end{aligned}$$

Furthermore the square of the type O' can be expressed in terms of the type O. Therefore S can be written

$$(2) \qquad S = R(O) + \sum O'R'(O),$$

where R and R' denote rational integral functions.

Every G' concomitant can be reduced to one of the forms

$$(3) \qquad R(O),\ \sum O'R'(O),$$

according as it is even or odd (§ 1).

Consider now the symbolic representation of the concomitants of double binary forms. Every concomitant of $a_\lambda^m a_\mu^n$ is expressible symbolically through the types: *

$$(ab),\ (a\beta),\ a_\lambda,\ a_\mu,\ (ar),\ (a\rho),$$

* If λ and μ are cogredient we must add the types (aa), a_μ, a_λ, $(a\rho)$, (ar); for distinct transformations of λ and μ, however, such types are not invariant. Cf. CAPELLI, Giornale di Matematiche, 1879, p. 71.

by the general theory of the Clebsch-Aronhold symbolism. If we limit ourselves to equi-forms $f = a_\lambda^n a_\mu^n$, every equi-concomitant B is symbolically a concomitant of the bilinear forms $a_\lambda a_\mu$, $b_\lambda b_\mu$, $\cdots$; and therefore (§ 6) expressible through the types

$$
\begin{aligned}
&\Omega: && [ab] = (ab)(a\beta),\ [au] = (ar)(a\rho),\ \{a\lambda\} = a_\lambda a_\mu\ ;\\
&\Omega': && [abcd] = \{(ac)(bd)(a\beta)(\gamma\delta) - (ab)(cd)(a\gamma)(\beta\delta)\}\ ,\\
&&& [abcu] = \{(ac)(a\beta)(br)(\gamma\rho) - (ab)(a\gamma)(cr)(\beta\rho)\}\ ,\\
&&& \{abc\lambda\} = \{(ac)(a\beta)\, b_\lambda\gamma_\rho - (ab)(a\gamma)\, c_\lambda\beta_\rho\}\ .
\end{aligned}
\tag{4}
$$

Of the second type Ω', only the first power is necessary (§ 7); therefore

$$B = R(\Omega) + \sum \Omega' R'(\Omega)\,. \tag{5}$$

Those B's which are invariant also for the transformations H, that is, concomitants of the group G', can be reduced to one of the forms:

$$R(\Omega),\quad \sum{}' \Omega' R'(\Omega), \tag{6}$$

*the first characterising those of even, and the second those of odd character.**

If now the forms $f = a_\lambda^n a_\mu^n$ and $F = F_x^n$ *correspond* in the sense of § 18, the types Ω, Ω' and O, O' are connected by the relations given in the preceding section. From (2), (3), (4), (5) we have then the fundamental

THEOREM.—*From every equi-concomitant of a system of double binary equi-forms, may be derived a concomitant of the system of corresponding (apolar) quaternary forms, by substituting for every type symbol Ω, Ω' a corresponding type symbol O, O'; conversely, a binary concomitant may be derived from every quaternary concomitant. In this transformation an even G' concomitant remains an even G' concomitant, and an odd remains odd. Geometrically, corresponding concomitants are equivalent, their vanishing having the same interpretation. The equivalence holds also in an algebraic sense, since corresponding forms are connected by the same syzygies, and a complete system remains complete after transformation.*

Another set of relations is obtained by considering, instead of the G and G' concomitants of the quaternary forms F, F', $\cdots$, the ordinary (projective) concomitants of the enlarged system:†

$$Q,\ F,\ F',\ \cdots.$$

* By the method employed in § 1, it may be shown that for G concomitants the factor produced by transformation is of the form $D_1^k D_2^l$, where $D_1 = (ad - bc)$, $D_2 = (a\delta - \beta\gamma)$, (4) § 10; and for G' concomitants the factor produced by the transformations G is of the form $D_1^k D_2^l$, while that produced by the transformations H may be either $D_1^k D_2^l$ or $-D_1^k D_2^l$, thus creating the distinction of even and odd G' concomitants.

† Here, and in the following relations, the coefficients p_{ik} of the fundamental quadric Q are *not* regarded as numbers, i. e., they are not included in the domain from which the coefficients of concomitants may be chosen, as was the case in the preceding.

From § 5 it may be shown that every such concomitant T, multiplied by a sufficiently high power of Δ, can be symbolically expressed in one and only one of the forms

$$R(\Delta, O), \quad \sum O' R'(\Delta, O), \tag{7}$$

the first form characterizing those of even, and the second those of odd weight. Employing the same symbolic transformation as in the preceding theorem, and comparing with (6), we obtain this result:

The equi-ordinal G' concomitants of a system of equi-forms

$$f = a_\lambda^n a_\mu^n, \quad f' = b_\lambda^m \beta_\mu^m, \cdots,$$

may be obtained from the ordinary concomitants of the fundamental quadric Q, and the corresponding apolar quaternary forms

$$F = A_x^n, \quad F' = B_x^m, \cdots.$$

The converse is also true.

Every G concomitant is the sum of two G' concomitants; for denoting by $\bar{G}$ the result of operating on the concomitant G by means of an improper transformation,* we have

$$G = \frac{G + \bar{G}}{2} + \frac{G - \bar{G}}{2}.$$

Therefore from every equi-concomitant of the binary forms, we can obtain in general two concomitants T. Complete systems stand in this relation:

The binary concomitants corresponding to a complete T system form a complete equi-binary system. The T concomitants obtained from a complete equi-binary system, together with the discriminant Δ, form a complete T system, provided Δ^{-1} is regarded as an integral invariant.

§ 22. *Examples of the transformation of concomitants.*—The Hessian of

$$F = A_x^n$$

is

$$(ABCD)^2 A_x^{n-2};$$

if we apply (3) and (6) of § 20, and the symbols defined in (1) § 21, this becomes (omitting powers of D)

$$[abcd]^2 \{a\lambda\}^{n-2}, \tag{1}$$

which is the corresponding binary form. Expanded,

* We may derive $\bar{G}$ from G by interchanging, in the symbolic representation of G, a with α and λ with μ.

$$(2)\quad [abcd]^2 = \begin{vmatrix} (ab)(a\beta), & (ac)(a\gamma), & (ad)(a\delta) \\ . & . & . \end{vmatrix}$$

$$= 2\{(ab)^2(cd)^2(a\beta)^2(\gamma\delta)^2 - (ab)(bc)(cd)(da)(a\beta)(\beta\gamma)(\gamma\delta)(\delta a)\},$$

$$\{a\lambda\}^{n-2} = a_\lambda^{n-2} a_\mu^{n-2}.$$

The Jacobian of four surfaces A_x^n, $B_x^{n_1}$, $C_x^{n_2}$, $D_x^{n_3}$,

$$J = (ABCD) A_x^{n-1} B_x^{n_1-1} C_x^{n_2-1} D_x^{n_3-1},$$

becomes
$$J' = [abcd]\{a\lambda\}^{n-1}\{b\lambda\}^{n_1-1}\{c\lambda\}^{n_2-1}\{d\lambda\}^{n_3-1},$$

which coincides with the covariant S_1 of § 14. The non-symbolic expressions are therefore

$$J = \begin{vmatrix} F_1 & F_2 & F_3 & F_4 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ . & . & . & . \\ . & . & . & . \end{vmatrix}, \quad J' = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ \phi_{11} & \phi_{12} & \phi_{21} & \phi_{22} \\ . & . & . & . \\ . & . & . & . \end{vmatrix}.$$

If one of the surfaces is the fundamental quadric Q, the Jacobian becomes $(ABCp) A_x^{n-1} B_x^{n_1-1} C_x^{n_2-1} p_x$, with the binary equivalent:

$$\{abc\lambda\}\{a\lambda\}^{n-1}\{b\lambda\}^{n_1-1}\{c\lambda\}^{n_2-1} = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ \phi_{11} & \phi_{12} & \phi_{21} & \phi_{22} \\ x_{11} & x_{12} & x_{21} & x_{22} \\ \lambda_2\mu_2 & -\lambda_2\mu_1 & -\lambda_1\mu_2 & \lambda_1\mu_1 \end{vmatrix},$$

which is the S_2 of § 14.

Consider now, as an example of the second kind, the problem of expressing binary transvectants in quaternary form. Let the binary forms be

$$f = a_\lambda^m a_\mu^n, \quad \phi = b_\lambda^n \beta_\mu^n,$$

and the corresponding quaternary,

$$F = A_x^m, \quad \phi = B_x^n.$$

The transvectant is

$$(f\phi)_{kk} = (ab)^k (a\beta)^k a_\lambda^{m-k} b_\lambda^{n-k} a_\mu^{m-k} \beta_\mu^{n-k}.$$

Therefore, applying (4) and (6) of § 20,

$$(3)\qquad D^{m+n+4k}(f\phi)_{kk} = A_p A_{p'} \cdots A_{p^{(k-1)}} B_p B_{p'} \cdots B_{p^{(k-1)}} A_x^{m-k} B_x^{n-k}.$$

§ 23. *The osculating circle in invariant form.*—As an application of the preceding methods, consider the problem of determining the osculating circle at a point of the curve $f \equiv a_\lambda^n a_\mu^n = 0$. Let the corresponding quaternary form be $F = A_x^n$; then by applying HESSE's formula * for the osculating plane to the space curve

$$F \equiv A_x^n = 0, \quad Q \equiv p_x^2 = 0,$$

the osculating circle C' at the point y_i is found to be

$$C' \equiv G(y)\sum F_i(y)x_i - H(y)\sum Q_i(y)x_i = 0, \tag{1}$$

where G and H are the covariants

$$G = \begin{vmatrix} |Q_{ik}| & F_i \\ F_k & 0 \end{vmatrix}, \quad H = \frac{1}{(n-1)^2}\begin{vmatrix} |F_{ik}| & Q_i \\ Q_k & 0 \end{vmatrix}.$$

To obtain the corresponding binary forms we must write these symbolically, the results being,

$$G = A_P B_P A_x^{n-1} B_x^{n-1},$$

$$H = \frac{1}{6(n-1)^2}(pABC)(p'ABC)p_x p_x' A_x^{n-2} B_x^{n-2} C_x^{n-2};$$

the binary forms are therefore,

$$g = (ff)_{11},$$

$$h = \frac{1}{6(n-1)^2}\{(ac)(a\beta)b_\lambda\gamma_\mu - (ab)(a\gamma)c_\lambda\beta_\mu\}^2 a_\lambda^{n-2} b_\lambda^{n-2} c_\lambda^{n-2} a_\mu^{n-2}\beta_\mu^{n-2}\gamma_\mu^{n-2}.$$

The osculating circle at the point $P = \xi, \eta$ is

$$C' \equiv g(\xi\eta)a_\xi^{n-1}a_\eta^{n-1}a_\lambda a_\mu - h(\xi\eta)(\xi\lambda)(\eta\mu) = 0. \tag{2}$$

This can be put into more convenient form by introducing the degenerate circle at P,

$$\bar{P} \equiv (\xi\lambda)(\eta\mu) = 0,$$

and the polar circle at P,

$$C \equiv a_\xi^{n-1}a_\eta^{n-1}a_\lambda a_\mu = 0;$$

equation (2) may then be written (omitting the arguments $\xi\eta$ in g and h),

$$C' = gC - h\bar{P}. \tag{3}$$

* HESSE, Crelle, vol. 41, p. 283; or SALMON-FIEDLER, *Raumgeometrie*, 3d edition, vol. II, page 156, line 1.

The invariant of C' is

$$I(C'C') = g^2 I(CC) - 2ghI(C\bar{P}) + h^2 I(\bar{P}\bar{P});$$

but

$$I(CC) = (ff)_{11} = g, \quad I(C\bar{P}) = f, \quad I(\bar{P}\bar{P}) = 0;$$

therefore

$$I(C'C') = g(g^2 - fh) = g^3,$$

since only the points ξ, η on the curve f are considered.

The osculating circle is degenerate only at the points where g cuts f, i. e., at the minimal points of f (§ 13).

If $h = 0$, C' coincides with C; if they coincide, either $h = 0$ or C is degenerate. Therefore:

The points of f where the osculating and polar circles coincide without degenerating, are cut out by the curve h; their number is $2n(3n - 4)$.

From the orders of g and h in ξ, η we have:

Through any point of the plane, pass $6n(n - 1)$ osculating circles of f.

§ 24. *The apolar* relation of curves.*—Consider any two $C_{n,n}$'s:

$$f \equiv a_\lambda^n a_\mu^n = 0, \quad f' \equiv b_\lambda^n \beta_\mu^n = 0.$$

Two C_{nn}'s are said to be harmonic, conjugate, or apolar when their bilinear invariant vanishes.

The bilinear invariant is (§ 13)

$$(1) \qquad (ff')_{nn} = (ab)^n(a\beta)^n = \sum_{h,k}(-1)^{h+k}\binom{n}{h}\binom{n}{k} a_{hk} b_{n-h,\,n-k}.$$

Let the corresponding surfaces be

$$(2) \qquad F \equiv A_x^n = 0, \quad F' \equiv B_x^n = 0,$$

and their reciprocals with respect to Q,

$$(3) \qquad \phi \equiv u_\rho^n = 0, \quad \phi' \equiv u_\sigma^n = 0.$$

The polar of the plane u_i has the coördinates $u_P P_i$; therefore ϕ and ϕ' may be written

$$\phi = A_P A_{P'} \cdots A_{P^{(n-1)}} u_P \cdots u_{P^{(n-1)}} = u_\rho^n,$$

$$\phi' = B_P B_{P'} \cdots B_{P^{(n-1)}} u_P \cdots u_{P^{(n-1)}} = u_\sigma^n;$$

from which

$$\rho_i = A_P P_i, \quad \sigma_i = B_P P_i.$$

* The apolarity considered relates of course to the geometries (A) or (B) of the introduction, and is distinct from ordinary (projective) apolarity: two plane curves which are inversionally apolar are not in general projectively apolar; and conversely, projectively apolar curves are not in general inversionally apolar.

But from (3) § 22,

$$(ff')_{nn} = A_P \cdots A_{P^{(n-1)}} B_P \cdots B_{P^{(n-1)}};$$

substituting the values of either ρ_i or σ_i, we have

$$(ff')_{nn} = (ab)^n(\alpha\beta)^n = A^n_\sigma = B^n_\rho. \tag{4}$$

The invariant A^n_σ, however, is the bilinear invariant of the surfaces F and ϕ', and B^n_ρ is the bilinear invariant of F' and ϕ. Therefore,

*If C_{nn} and C'_{nn} are harmonic, so are the surface corresponding to either, and the reciprocal with respect to Q of the surface corresponding to the other.** [Added Sept. 28, 1900. More generally, *every* surface of the nth order through the one curve, is apolar to the reciprocal of the surface corresponding to the order.]

An interpretation of the apolar relation of C_{nn} and C'_{nn} can also be obtained without passing to space. With respect to the curve $f = 0$, every circle $r_\lambda \rho_\mu$ has a polar circle $(ar)(a\rho)\cdots(ar^{(n-2)})(a\rho^{(n-2)})a_\lambda a_\mu = 0$; from the binary form of ϕ,

$$\phi = (ar)(a\rho)\cdots(ar^{(n-1)})(a\rho^{(n-1)}), \tag{5}$$

we see then that ϕ can be defined as the complex of circles which are orthogonal to their polar circles with respect to f. Similarly ϕ' is the complex of circles which are orthogonal to their polar circles with respect to f'. The curves f and ϕ stand in the following reciprocal relation: f is the locus of the points contained in ϕ, and ϕ is the complex of circles which are orthogonal to their polar circles with respect to f. If f and f' are harmonic, so are the forms F and ϕ', and F' and ϕ.

If f and f' are cyclics, the bilinear invariant is

$$\begin{aligned}(ff')_{22} = {} & a_{00}b_{22} - 2a_{01}b_{21} + a_{02}b_{20} - 2a_{10}b_{12} + 4a_{11}b_{11} \\ & + a_{22}b_{00} - 2a_{21}b_{01} + a_{20}b_{02} - 2a_{12}b_{10}.\end{aligned} \tag{6}$$

When the cyclics f and f' are apolar, the quadric ϕ is inscribed in an infinite number of tetrahedra which are self-polar with respect to F' and a similar relation exists between ϕ' and F.

In the plane the relation may be given by introducing the notion of circles which are conjugate with respect to a cyclic. The polar circle of any circle C with respect to f, is the locus of points whose polar circles are orthogonal to C; two circles are conjugate when each is orthogonal to the polar circle of the other. If then we call a set of four circles which are mutually conjugate a self-conjugate set we have:

If two cyclics f, f' are apolar, the complex ϕ of circles orthogonal to their polar circles with respect to f, contains an infinity of sets which are self-conjugate with

* A corresponding interpretation of the conjugacy of two *simple* binary forms is given by SCHLESINGER, *Ueber conjugirte binäre Formen und deren geometrische Construction*, Mathematische Annalen, vol. 22, pp. 520–568, 1883.

respect to f'; and similarly with ϕ' and f. Conversely, if ϕ and f' or ϕ' and f stand in this relation, f and f' are apolar.

The importance of this theorem lies in the fact that it will serve as a basis for a synthetic theory of linear systems of cyclics, similar to that which has been developed for conics:* for a linear ν-membered system of cyclics,

$$t_1 f_1 + t_2 f_2 + \cdots + t_\nu f_\nu = 0,$$

may be defined as the totality of cyclics which are apolar to $9 - \nu$ linearly independent fixed cyclics.

§ 25. *Quaternary methods for the general double binary form.*—For the quaternary study of forms

$$f = a_\lambda^{n+\kappa} a_\mu^n,$$

whose partial orders are different, we may pass from f to an equi-concomitant. Thus we may consider

$$(ff)_{\kappa 0} = (ab)^\kappa a_\lambda^n a_\mu^n.$$

The concomitants of this form may be studied by quaternary methods; but they will not, of course, include all the concomitants of f.

A system of equi-forms which is equivalent to f is given by any one of the systems of polars:

$$P_0 = a_\xi^\kappa a_\lambda^n a_\mu^n \quad \text{or} \quad P_h = a_\xi^{\kappa+h} a_\eta^h a_\lambda^{n-h} a_\mu^{n-h} \quad (h = 1, 2, \cdots, n-1).$$

A quaternary method of treating f consists then in the study of the $(1, 1)$ correspondence, defined by $P_h = 0$, between the points $(\xi\eta)$ of Q, and the two-parameter (non-linear) system of surfaces of order $n - h$, which corresponds, by § 18, to $P_h = 0$; or, in the study of the $(1, 1)$ correspondence, defined by $P_0 = 0$, between the generators $\lambda = \xi$ of one series on the quadric Q, and the one-parameter system of surfaces of order n which corresponds to $P_0 = 0$.

CHAPTER V.

The Cyclic Curves.

In tetracyclic coördinates the cyclic is represented by the quaternary quadric form a_x^2, and in minimal coördinates by the quadri-quadric double binary form $a_\lambda^2 a_\mu^2$. The first representation brings the inversion theory of the cyclic into relation with the theory of two quadric surfaces, while the second connects it with the general (2, 2) correspondence. The relation of the two points of view is most easily grasped by passing from the plane to the isomorphic problem in

* Darboux, Bulletin des Sciences Mathématiques, vol. 1. Reye, *Ueber Systeme und Gewebe von Kegelschnitten*, Crelle, vol. 82, pp. 54–83, 1877.

space of a bi-quadratic curve of the first species on a fixed quadric Q; for such a curve may be determined either by cutting Q with another quadric or by establishing a $(2, 2)$ correspondence between the two systems of generators on Q. The previous methods will now be applied to the discussion, from both points of view, of the concomitants of the cyclic—their complete systems, relations and geometric interpretations.*

§ 26. *System of two quadrics.*—Let the identity be $U' \equiv \sum p_{ik}x_ix_k \equiv p_x^2 = 0$, and the equation of the cyclic

$$U = \sum a_{ik}x_ix_k = a_x^2 = 0;$$

then (§ 4) the concomitants of the curve may be derived from the projective concomitants of the two quaternary quadrics U', U. Denoting the corresponding class quadrics by

$$\sigma' = u_P^2, \ \sigma = u_a^2,$$

the invariants are

$$\Delta' = \tfrac{1}{2}\tfrac{1}{4}(pp'p''p''')^2 = \tfrac{1}{4}p_P^2, \quad \Delta = \tfrac{1}{2}\tfrac{1}{4}(abcd)^2 = \tfrac{1}{4}a_a^2,$$

$$\theta' = \tfrac{1}{6}(pp'p''a)^2 = a_P^2, \qquad \theta = \tfrac{1}{6}(pabc)^2 = p_a^2,$$

$$\phi = \tfrac{1}{4}(pp'ab)^2;$$

the covariants are

$$U = a_x^2, \qquad U' = p_x^2,$$

$$S' = a_Pb_Pa_xb_x, \qquad S = p_ap'_ap_xp'_x,$$

$$Y' = \tfrac{1}{4}(abpp')b_Pc_Pp'_ap''_ca_xc_xp_xp''_x;$$

and the contravariants are

$$\sigma = u_a^2 = \tfrac{1}{6}(abcu)^2, \quad \sigma' = u_P^2 = \tfrac{1}{6}(pp'p'''u)^2,$$

$$\tau' = \tfrac{1}{2}(pp'au)^2, \qquad \tau = \tfrac{1}{2}(pabu)^2,$$

$$\omega = \tfrac{1}{4}u_Pu_a(pp'au)(p''bcu)\begin{vmatrix} p_a & p'_a & a_a \\ p_P & p'_P & a_P \\ (pp''bc) & (p'p''bc) & (ap''bc) \end{vmatrix}.$$

* A non-invariant discussion of these cyclic curves or bi-circular quartics is given in DARBOUX, *Sur une classe remarquable des courbes*, etc., Paris, 1872, reprint 1896.

In terms of these fifteen forms: *

$$(\Sigma_1)\qquad \begin{matrix} \Delta, & \Delta', & \theta, & \theta', & \phi, \\ U, & U', & S, & S', & Y', \\ \sigma, & \sigma', & \tau, & \tau', & \omega, \end{matrix}$$

every concomitant of U, U' involving at most one series of variables, can be rationally and integrally expressed.

Referred to their common self-polar tetrahedron, the quadrics may be written

$$U' = \sum x_i^2, \quad U = \sum a_i x_i^2;$$

and the complete system $(\sum_1)$ takes the normal form:

$$\begin{aligned}
&\Delta' = 1, \quad \theta' = \sum a_i, \quad \phi = \sum a_i a_k, \quad \theta = \sum a_i a_k a_l, \quad \Delta = a_1 a_2 a_3 a_4;\\
&U' = \sum x_i^2, \quad U = \sum a_i x_i^2, \quad S' = \sum a_i^2 x_i^2, \quad S = \sum a_k a_l a_m x_i^2,\\
&Y' = (a_1 - a_2)(a_1 - a_3)(a_1 - a_4)(a_3 - a_4)(a_4 - a_1)(a_2 - a_3) x_1 x_2 x_3 x_4;\\
(1)\qquad &\sigma' = \sum u_i^2, \quad \tau' = \sum (a_k + a_l + a_m) u_i^2,\\
&\tau = \sum (a_k a_l + a_l a_m + a_m a_k) u_i^2, \quad \sigma = \sum a_k a_l a_m u_i^2,\\
&\omega = (a_1 - a_2)(a_1 - a_3)(a_1 - a_4)(a_3 - a_4)(a_4 - a_1)(a_2 - a_3) u_1 u_2 u_3 u_4.
\end{aligned}$$

§27. *Quaternary system of the cyclic.*—According to the method of §4, to apply the above results to the cyclic, we must replace U by that member F, of the pencil $U + kU'$, which is apolar to U'; this apolar quadric is [(6) or (7), §3]

$$(1)\qquad F = A_x^2 = \Delta' U - \frac{\theta'}{4} U'.$$

The complete quaternary system of the curve consists then of the "identical" form

$$\sigma' = u_P^2$$

(so called since it is the same for all curves), *and of the proper concomitants:* †

* A non-symbolic discussion of these forms is given in SALMON-FIEDLER, *Raumgeometrie*, vol. I, Chapter XI. I have not thought it necessary to give the symbolic calculations by which the above results are obtained, as they are similar to those which occur in the corresponding ternary problem; cf. GORDAN'S system, CLEBSCH-LINDEMANN, *Geometrie*, vol. I, p. 291. The completeness of (Σ_1) may be proved by comparing with MERTEN'S equivalent system, Wiener Berichte, 1889, 2 A, pp. 733–8.

† There is no invariant corresponding to θ' of Σ_1, since in the present case $A_P^2 = 0$.

$$I=\tfrac{1}{4}(pp'AB)^2,\quad J=\tfrac{1}{6}(pABC)^2,\quad k=\tfrac{1}{24}(ABCD)^2;$$

$$F=A_x^2,\quad G=A_PB_PA_xB_x,\quad H=p_{\bar{A}}p'_{\bar{A}}p_xp'_x,$$

$$(2)\qquad Y=\tfrac{1}{4}(ABpp')B_PC_Pp'_{\bar{A}}p''_{\bar{A}}A_xC_xp_xp''_x;$$

$$\phi=\tfrac{1}{2}(pp'Au)^2,\quad \psi=\tfrac{1}{2}(pABu)^2,\quad \chi=u_{\bar{A}}^2,$$

$$Z=\tfrac{1}{4}u_Pu_{\bar{A}}(pp'Au)(p''BCu)\begin{vmatrix} p_{\bar{A}} & p'_{\bar{A}} & A_{\bar{A}} \\ p_P & p'_P & A_P \\ (pp''BC) & (p'p''BC) & (Ap''BC) \end{vmatrix}.$$

These concomitants:

$$(\Sigma_2)\qquad \begin{matrix} I,\ J,\ K, \\ F,\ G,\ H,\ Y, \\ \phi,\ \psi,\ \chi,\ Z,\ \sigma' \end{matrix}$$

are also, by (1), concomitants of U' and U; it follows that (Σ_2) is expressible integrally through (Σ_1). To obtain the explicit relations, it is sufficient to consider both systems (Σ_1) and (Σ_2) in their normal forms; the normal form of Σ_1 has been given in § 26, and similarly the normal form of Σ_2 (omitting the identical form) is

$$I=\sum\beta_i\beta_k,\quad J=\sum\beta_i\beta_k\beta_l,\qquad K=\beta_1\beta_2\beta_3\beta_4,$$

$$F=\sum\beta_ix_i^2,\quad G=\sum\beta_i^2x_i^2,\qquad H=\sum\beta_k\beta_l\beta_mx_i^2,$$

$$(3)\qquad Y=(\beta_1-\beta_2)\cdots(\beta_3-\beta_4)x_1x_2x_3x_4;$$

$$\phi=-\sum\beta_iu_i^2,\quad \psi=\sum(I+\beta_i^2)u_i^2,\quad \chi=\sum\beta_k\beta_l\beta_mu_i^2,$$

$$Z=(\beta_1-\beta_2)\cdots(\beta_3-\beta_4)u_1u_2u_3u_4.$$

The α's and β's are connected, from (1), by the relation:

$$\beta_i=\Delta'\alpha_i-\frac{\theta'}{4},$$

so that $\sum\beta_i=0$. Introducing the value of β_i in the above normal forms, and comparing the results with (1) § 26, we obtain, after some calculation, the required relations between Σ_1 and Σ_2 as follows:

$$8I=8\Delta'^2\phi-3\Delta'\theta'^2,$$

$$16J=16\Delta'^3\theta-8\Delta'^2\theta'\phi+2\Delta'\theta'^3,$$

$$(4)\qquad 256K=256\Delta'^4\Delta-64\Delta'^3\theta'\theta-16\Delta'^2\theta'^2\phi-3\Delta'\theta'^4,$$

$$4F=4\Delta'U-\theta'U',$$

$$16G=16\Delta'^2S'-8\Delta'\theta'U+\theta'^2U',$$

$$
\begin{aligned}
64H &= 64\Delta'^3S + 12\Delta'\theta'^2U - 16\Delta'^2\theta'S' + (3\theta'^3 - 16\theta\,\phi)U', \\
Y &= \Delta'^4Y', \\
4\phi &= 4\Delta'\tau' - 3\theta'\sigma', \\
16\psi &= 16\Delta'^2\tau - 8\Delta'\theta'\tau' + 3\theta'^2\sigma', \\
64\chi &= 64\Delta'^3\sigma - \Delta'^2\theta'\tau + 4\Delta'\theta'^2\tau' + \theta'^3\sigma', \\
Z &= \Delta'^4\omega\,.
\end{aligned} \tag{4}
$$

The formulæ (2) give the concomitants of the cyclic in terms of the identity $U' = 0$ and the apolar form F, while the formulæ (4) give them in terms of the identity $U' = 0$ and the most general quadric U which can represent the curve.

It is convenient, for some purposes, to replace the covariants of the system $\sum_2$ by the corresponding apolar forms: *

$$
F,\ G_1 = G + \frac{I}{2}U',\ \ H_1 = H - \frac{J}{4}U',\ \ Y. \tag{5}
$$

§28. *Binary system of equi-forms.*—The quaternary system $\sum_2$ of the preceding article will now be transformed so as to apply to the cyclic represented by the double binary form

$$
f = a_\lambda^2 a_\mu^2\,,
$$

the method being that developed in §21.

For the invariants, the formulæ (5) §20 give †

$$
4I = (pp'AB)^2 = 2\begin{vmatrix} (aa)(a\alpha) & (ab)(a\beta) \\ (ab)(a\beta) & (bb)(\beta\beta) \end{vmatrix} = -\,2(ab)^2(a\beta)^2;
$$

$$
\begin{aligned}
6J = (pABC)^2 &= -\begin{vmatrix} 0 & (ab)(a\beta) & (bc)(a\gamma) \\ (ba)(\beta a) & 0 & (bc)(\beta\gamma) \\ (ca)(\gamma a) & (cb)(\gamma\beta) & 0 \end{vmatrix} \\
&= -\,2(ab)(bc)(ca)(a\beta)(\beta\gamma)(\gamma a)\,;
\end{aligned}
$$

$$
\begin{aligned}
24K = (ABCD)^2 &= \{(ac)(bd)(a\beta)(\gamma\delta) - (ab)(cd)(a\gamma)(\beta\delta)\} \\
&= 3(ab)^2(cd)^2(a\beta)^2(\gamma\delta)^2 - 6(ab)(bc)(cd)(da)(a\beta)(\beta\gamma)(\gamma\delta)(\delta a) \\
&= 2(ab)^2(cd)^2(a\gamma)^2(\beta\delta)^2 - 6(ab)(bd)(ac)(cd)(a\beta)(\beta\delta)(a\gamma)(\gamma\delta)\,.
\end{aligned}
$$

* That Y is apolar to U', is a special case of the easily proved theorem: the necessary and sufficient condition in order that the product of a number of linear forms shall be apolar to a quadric, is that each linear form shall be conjugate to every other with respect to the quadric. It will appear later (§30) that Y satisfies this condition.

† The D which appears in §20 is here taken as unity for convenience.

Denoting by f, g, h the covariants corresponding to F, G, H, we have, by (5), (6) § 20,

$$f = a_\lambda^2 a_\mu^2, \quad g = (ab)(a\beta)a_\lambda b_\lambda a_\mu \beta_\mu,$$
$$h = \{(ac)(a\beta)b_\lambda \gamma_\mu - (ab)(a\gamma)c_\lambda \beta_\mu\}^2;$$

while Y, the Jacobian of $U'FGH$, becomes (§ 22)

$$Y = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ g_{11} & g_{12} & g_{21} & g_{22} \\ h_{11} & h_{12} & h_{21} & h_{22} \\ \lambda_2\mu_2 & -\lambda_2\mu_1 & -\lambda_1\mu_2 & \lambda_1\mu_1 \end{vmatrix}.$$

The contravariants are

$\sigma' = (rr')(\rho\rho')$,

$\phi = -(ar)(ar')(a\rho)(a\rho')$,

$\psi = -(ab)(a\beta)(ar)(br')(a\rho)(\beta\rho')$,

$\chi = \{(ac)(a\beta)(br)(\gamma\rho) - (ab)(a\gamma)(cr)(\beta\rho)\}\{(ac)(a\beta)(br')(\gamma\rho') - (ab)(a\gamma)(cr')(\beta\rho')\}$,

$Z =$ Jacobian $(\sigma'\phi\psi\chi)$.

Thus the *equi-binary system of the cyclic*, or the complete system of invariants, contravariants and equi-ordinal covariants of the quadri-quadric

$$f = a_\lambda^2 a_\mu^2,$$

consists of the forms:

$$(\Sigma_3) \qquad \begin{matrix} I, \ J, \ K, \\ f, \ g, \ h, \ Y, \\ \phi, \ \psi, \ \chi, \ Z, \ \sigma'. \end{matrix}$$

Of these σ', since it does not depend upon f, may be called the identical contravariant.

§ 29. *Non-symbolic values, and normal form of* Σ_3.—In expanded form f is

$$(1) \qquad \begin{aligned} f = \ & (a_{00}\mu_1^2 + 2a_{01}\mu_1\mu_2 + a_{02}\mu_2^2)\lambda_1^2 \\ & + 2(a_{10}\mu_1^2 + 2a_{11}\mu_1\mu_2 + a_{12}\mu_2^2)\lambda_1\lambda_2 \\ & + (a_{20}\mu_1^2 + 2a_{21}\mu_1\mu_2 + a_{22}\mu_2^2)\lambda_2^2. \end{aligned}$$

If we take for the fundamental quadric,

$$(2) \qquad U' = 2x_1x_4 - 2x_2x_3,$$

the transformation formulæ may be written

$$(3) \qquad x_1 = \lambda_1\mu_1, \quad x_2 = \lambda_1\mu_2, \quad x_3 = \lambda_2\mu_1, \quad x_4 = \lambda_2\mu_2,$$

and the quadric corresponding to f is (§ 18)

$$F = \sum A_{ik} x_i x_k ,$$

where

$$\begin{aligned} &A_{11} = a_{00}, \quad A_{12} = a_{01}, \quad A_{13} = a_{10}, \quad A_{14} = a_{11}, \quad A_{22} = a_{02}, \\ &A_{23} = a_{11}, \quad A_{24} = a_{12}, \quad A_{33} = a_{20}, \quad A_{34} = a_{21}, \quad A_{44} = a_{22}. \end{aligned} \tag{4}$$

The concomitants $\sum_3$ of f may then be obtained directly from the concomitants of U', F. The invariants are obtained by forming the discriminant of $U' + \rho F$:

$$D(\rho) = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14}+\rho \\ A_{21} & A_{22} & A_{23}-\rho & A_{24} \\ A_{31} & A_{32}-\rho & A_{33} & A_{34} \\ A_{41}+\rho & A_{42} & A_{43} & A_{44} \end{vmatrix} = \begin{vmatrix} a_{00} & a_{01} & a_{10} & a_{11}+\rho \\ a_{01} & a_{02} & a_{11}-\rho & a_{12} \\ a_{10} & a_{11}-\rho & a_{20} & a_{21} \\ a_{11}+\rho & a_{12} & a_{21} & a_{22} \end{vmatrix} \tag{5}$$

$$= \rho^4 + I\rho^2 + J\rho + K .$$

The expansion gives

$$\begin{aligned} I &= 2(a_{10}a_{12} + a_{01}a_{21} - a_{11}^2) - a_{00}a_{22} - a_{02}a_{20}, \\ J &= 2 \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}, \quad K = \begin{vmatrix} a_{00} & a_{01} & a_{10} & a_{11} \\ a_{01} & a_{02} & a_{11} & a_{12} \\ a_{10} & a_{11} & a_{20} & a_{21} \\ a_{11} & a_{12} & a_{21} & a_{22} \end{vmatrix}. \end{aligned} \tag{6}$$

Similarly, the covariants may be obtained from

$$\begin{vmatrix} 0 & \lambda_2\mu_2 & -\lambda_2\mu_1 & -\lambda_1\mu_2 & \lambda_1\mu_1 \\ \lambda_2\mu_2 & a_{00} & a_{01} & a_{10} & a_{11}+\rho \\ -\lambda_2\mu_1 & a_{01} & a_{02} & a_{11}-\rho & a_{12} \\ -\lambda_1\mu_2 & a_{10} & a_{11}-\rho & a_{20} & a_{21} \\ \lambda_1\mu_1 & a_{11}+\rho & a_{12} & a_{21} & a_{22} \end{vmatrix} = \rho^2 f - \rho g - h , \tag{7}$$

and the contravariants from

$$\begin{vmatrix} 0 & r_{11} & r_{12} & r_{21} & r_{22} \\ r_{11} & a_{00} & a_{01} & a_{10} & a_{11}+\rho \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \rho^3\sigma' + \rho^2\phi + \rho\psi + \chi . \tag{8}$$

It will suffice, for later applications, to write down the explicit values of the covariants and contravariants only for a certain normal form which will now be considered. The normal quaternary form of the cyclic is

$$[\mathrm{A}] \qquad \begin{cases} U' = \sum y_i^2, \\ F = \sum \beta_i y_i^2, \end{cases} \qquad (\Sigma\beta_i = 0).$$

This, by the substitution:

$$(9) \qquad \begin{aligned} \sqrt{2}y_1 &= x_1 + x_4, & \sqrt{2}x_1 &= y_1 + cy_4, \\ i\sqrt{2}y_2 &= x_2 + x_3, & \sqrt{2}x_2 &= y_3 + cy_2, \\ \sqrt{2}y_3 &= x_2 - x_3, & \sqrt{2}x_3 &= y_3 - cy_2, \\ i\sqrt{2}y_4 &= x_1 - x_4, & \sqrt{2}x_4 &= y_1 - cy_4, \end{aligned}$$

becomes

$$[\mathrm{B}] \qquad \begin{cases} U' = 2x_1x_4 - 2x_2x_3, \\ F = k(x_1^2 + x_4^2) + l(x_2^2 + x_3^2) + 2m(x_1x_4 + x_2x_3), \end{cases}$$

where

$$(10) \qquad \begin{aligned} 2k &= \beta_1 - \beta_4, & \beta_1 &= m + k, \\ 2l &= \beta_3 - \beta_2, & \beta_2 &= -m - l, \\ 2m &= \beta_1 + \beta_4, & \beta_3 &= -m + l, \\ &= -\beta_2 - \beta_3, & \beta_4 &= m - k. \end{aligned}$$

Finally, by (3),

$$[\mathrm{C}] \qquad f = k\lambda_1^2\mu_1^2 + l\lambda_1^2\mu_2^2 + 4m\lambda_1\mu_1\lambda_2\mu_2 + l\mu_1^2\lambda_2^2 + k\lambda_2^2\mu_2^2.$$

The general cyclic can be reduced to any of the normal forms [A], [B], [C]; each contains two essential constants as it should, since the cyclic has $8 - 6 = 2$ absolute invariants.

For the binary normal form [C], the system $\sum_3$ is

$$(11) \qquad \begin{aligned} &I = -(k^2 + l^2 + 2m^2), \quad J = 2m(k^2 - l^2), \quad K = (m^2 - k^2)(m^2 - l^2); \\ &f = k(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) + l(\lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2) + 4m\lambda_1\lambda_2\mu_1\mu_2, \\ &g = 2km(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) - 2lm(\lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2) + 2(k^2 - l^2)\lambda_1\lambda_2\mu_1\mu_2, \\ &h = k(l^2 - m^2)(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) + l(k^2 - m^2)(\lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2) \\ &\qquad\qquad - 2m(k^2 + l^2 - 2m^2)\lambda_1\lambda_2\mu_1\mu_2, \\ &Y = -\tfrac{1}{4}\sqrt{\delta}(\lambda_1^2\mu_1^2 - \lambda_2^2\mu_2^2)(\lambda_1^2\mu_2^2 - \lambda_2^2\mu_1^2); \end{aligned}$$

$$
\begin{aligned}
&\sigma' = 2r_{11}r_{22} - 2r_{12}r_{21}\,,\\
&\phi = -\,k(r_{11}^2 + r_{22}^2) - l(r_{12}^2 + r_{21}^2) - 2m(r_{11}r_{22} + r_{12}r_{21})\,,\\
&\psi = 2km(r_{11}^2 + r_{22}^2) - 2lm(r_{12}^2 + r_{21}^2) - 2(l^2 + m^2)r_{11}r_{22} + 2(k^2 + m^2)r_{12}r_{21}\,,\\
&\chi = k(l^2 - m^2)(r_{11}^2 + r_{22}^2) + l(k^2 - m^2)(r_{12}^2 + r_{21})^2\\
&\qquad\qquad - 2m(l^2 - m^2)r_{11}r_{22} - 2m(k^2 - m^2)r_{12}r_{21}\,,\\
&Z = -\,\tfrac{1}{4}\sqrt{\delta}\,(r_{11}^2 - r_{22}^2)(r_{12}^2 - r_{21}^2)\,,
\end{aligned}
\tag{11}
$$

where

$$\sqrt{\delta} = 4kl\{(k^2 + l^2)^2 - 16m^4\}\,.$$

§ 30. *The polars and the covariants.*—If we apply the definition of § 20 to the cyclic

$$f = a_\lambda^2 a_\mu^2\,,$$

we see that every point $P = (\xi,\ \eta)$ has five polars Q_{10}, Q_{01}, Q_{11}, Q_{12}, Q_{21}, whose expressions and geometric definitions follow.

The polar minimal lines,

$$L' = a_\xi a_\eta^2 a_\lambda = 0\,,\quad M' = a_\xi^2 a_\eta a_\mu = 0$$

are the harmonics of ξ (or η) with respect to the λ-lines (or μ-lines) which correspond to η (or ξ) in f. The polar circle

$$C' = a_\xi a_\eta a_\lambda a_\mu = 0\,,$$

is generated by the following correspondence: to each μ corresponds the harmonic of ξ with respect to that pair of λ's, to each of which corresponds in f a pair of μ's harmonic to η and μ; and to each λ, etc. The polar circle, then, cuts each of the minimal lines through P in the harmonic of P with respect to the intersections of the minimal line and the cyclic; and the conjugate point P', of P with respect to f, which is defined as the intersection of L' and M', is the inverse of P with respect to C'. The polars,

$$Q_{21} = a_\eta a_\lambda^2 a_\mu\,,\quad Q_{12} = a_\xi a_\lambda a_\mu^2\,,$$

are the loci of points whose polar μ-lines and λ-lines, respectively, pass through P; their intersections give the five points whose conjugates coincide with P.

It is convenient in the discussion of the relation of the polars and covariants to consider the cyclic in its quaternary normal form [A] § 29. If the coördinates of P are x_i,

$$
\begin{aligned}
&\text{the coördinates of } C' \text{ are } && \beta_i x_i\,,\\
&\text{“ } \quad \text{“ } \quad \text{“ } C'' \text{ “ } && \beta_k\beta_l\beta_m x_i\,,\\
&\text{and “ } \quad \text{“ } \quad \text{“ } P' \text{ “ } && (2\beta_i F - G)x_i\,,
\end{aligned}
\tag{1}
$$

where C'' is the "anti-polar" circle of P, i. e., the circle orthogonal to the polar circles of all points on the minimal lines through P. The polar and anti-polar circles of any point are orthogonal; for

$$I(C'C'') = \sum \beta_1\beta_2\beta_3\beta_4 x_i^2 = 0 .$$

The invariants of P, C', C'', P'' are

$$\begin{gathered} I(PC') = F, \quad I(PC'') = H, \quad I(C'C') = G, \quad I(C'C'') = 0 , \\ I(C'P') = FG, \quad I(C''C'') = JH - KG, \quad I(P'C'') = -GH, \end{gathered} \tag{2}$$

as may be proved by (3) § 27. From these we obtain the following interpretations.

The curve $F = 0$ is the given cyclic; its points are characterized by the fact that the polar circle of each passes through it.

The curve $G = 0$ is the locus of a point whose polar circle with respect to F is degenerate; it is also the locus of the vertices of degenerate apolar circles.

Finally, the locus of a point whose anti-polar circle passes through it, is $H = 0$.

If P, C', C'' have a common point, then

$$\begin{vmatrix} 0 & F & H \\ F & G & 0 \\ H & 0 & JH - KG \end{vmatrix} = F^2(KG - JH) - G^2H = 0 ;$$

and if the polar and anti-polar circles are tangent,

$$T(KG - JH) = 0 .$$

The conjugate of the conjugate of P has the coördinates:

$$[2\beta_i\{ + F^2(IF + H) + 3FG^2\} + 4F^2(JF + IG) + 4FG(IF + G) + G^3]x_i ,$$

so that the locus of a point whose second conjugate lies on the same minimal line is

$$2F^2\{4F(IF + H) + 3G^2\} = 0 .$$

Consider now the relations of the polar circles of a point with respect to F, G, H; these circles (§ 19) are to be found from the apolar forms of the covariants as given in (5) § 27. The coördinates are as follows:

$$\begin{gathered} P : \quad x_i , \quad C_G : \quad \left(\beta_i^2 + \frac{I}{2}\right)x_i , \\ C_F : \quad \beta_i x_i , \quad C_H : \quad \left(\beta_i\beta_j\beta_m - \frac{J}{4}\right)x_i . \end{gathered} \tag{4}$$

Their invariants are

$$I(PC_F) = F, \quad I(C_FC_G) = -\frac{I}{2}F - H, \quad I(C_FC_H) = -\frac{J}{4}F,$$

$$(5) \qquad I(C_GC_G) = JF, \quad I(C_GC_H) = KF - \frac{J}{4}G + \frac{I}{2}H,$$

$$I(C_HC_H) = \frac{J}{2}H - KG.$$

From these we obtain the (1, 1) transvectants of the corresponding double binary forms f, g, h [(1) §14]:

$$(ff)_{11} = I(C_fC_f) = g, \quad (fg)_{11} = -\frac{I}{2}f - h, \quad (fh)_{11} = -\frac{J}{4}f,$$

$$(6)$$

$$(gg)_{11} = Jf, \quad (gh)_{11} = Kf - \frac{J}{4}g - \frac{I}{2}h, \quad (hh)_{11} = \frac{J}{2}h - Kg;$$

and the substitution of these values in (2) §14 gives the reduction of Y^2 as follows:

$$Y^2 = \begin{vmatrix} g & -\frac{I}{2}f - h & -\frac{J}{4}f & f \\ -\frac{I}{2}f - h & Jf & Kf - \frac{J}{4}g - \frac{I}{2}h & g \\ -\frac{J}{4}f & Kf - \frac{J}{4}g - \frac{I}{2}h & \frac{J}{2}h - Kg & h \\ f & g & h & 0 \end{vmatrix}.$$

From §14 and the normal form (3) §27, we have:

The curve $Y = 0$ is the locus of a point which lies on the orthogonal circle of its polar circles with respect to f, g and h; this locus consists of four mutually orthogonal circles, the "director" circles of the cyclic. *

§ 31. *The discriminants.*—As defined in § 15, the discriminants are

$$D_1(\lambda) = \tfrac{1}{2}(ff)_{20} = \tfrac{1}{2}(a\beta)^2 a_\lambda^2 b_\lambda^2,$$
$$D_2(\mu) = \tfrac{1}{2}(ff)_{02} = \tfrac{1}{2}(ab)^2 a_\mu^2 \beta_\mu^2;$$

and for the normal form [C] § 29 they are

$$D_1(\lambda) = -kl\lambda^4 + (4m^2 - k^2 - l^2)\lambda^2 - kl,$$

$$(1)$$

$$D_2(\mu) = -kl\mu^4 + (4m^2 - k^2 - l^2)\mu^2 - kl.$$

* The covariant curves cut f in the following points: g, in the 8 minimal points (§13); h, in the 8 points where the polar circle is also the osculating circle (§23); Y, in the 16 points where the osculating circle has 4-point contact (CLEBSCH, Crelle, vol. 63, p. 9, gives the space theorem corresponding to the last result).

Calculating the invariants i, j of these biquadratics and comparing with the values of I, J, K in (11) § 29, we find

$$(2) \qquad 6i_1 = 6i_2 = I^2 + 12K, \quad 36j_1 = 36j_2 = -I^3 + 36IK - 6J^2;$$

these are also the invariants of

$$D(\rho) = \rho^4 + I\rho^2 + J\rho + K,$$

where $D(\rho)$ is equal to the characteristic determinant of the two quadrics U', F. This gives FROBENIUS' generalization of CAYLEY's theorem:*

The discriminants D_1, D_2 and the characteristic biquadratic D, have equal invariants.

Equated to 0, $D_1(\lambda)$ and $D_2(\mu)$ represent the four λ-tangents and the four μ-tangents of the cyclic; Cayley's theorem then states that these two sets of lines have equal anharmonic ratios.

The minimal points of f, i. e., the points of contact of the minimal tangents $D_1(\lambda) = 0$, $D_2(\mu) = 0$, are cut out (§ 13) by the (1, 1) transvectant

$$(ff)_{11} = g = 0;$$

but from (6) § 30,

$$(gg)_{11} = Jf;$$

therefore the minimal points of f are also the minimal points of g. The discriminants of g:

$$E_1(\lambda) = \tfrac{1}{2}(gg)_{20}, \quad E_2(\mu) = \tfrac{1}{2}(gg)_{02},$$

have then the interpretation: $E_1(\lambda) = 0$ represents the λ-lines through the points of contact of the μ-tangents $D_2(\mu) = 0$; and $E_2(\mu) = 0$, the μ-lines through the points of contact of the λ-tangents $D_1(\lambda) = 0$.

The normal form $[C]$ of E_1 is

$$(3) \qquad E_1(\lambda) = 4klm^2\lambda^4 + \{(k^2 - l^2)^2 - 4m^2(k^2 + l^2)\}\lambda^2 + 4klm^2;$$

comparing this with D_1 and its Hessian H_{D_1}, we obtain the following for E_1 (and similarly for E_2):

$$(4) \qquad E_1 = \frac{2I}{3} D_1 - 2H_{D_1}, \quad E_2 = \frac{2I}{3} D_2 - 2H_{D_2};$$

it follows that the quadruples of λ-lines represented by D_1 and E_1 are not independent, but connected by the fact that the sextic covariants of both represent the same six lines.†

*CAYLEY's theorem states the equality of the invariants of D_1, D_2; CAYLEY, Quarterly Journal, vol. 11, p. 83-91; other proofs have been given by CAPELLI, ZEUTHEN, LEPAIGE; FROBENIUS' generalization, Crelle's Journal, vol. 106, p. 129, 1890.

†CLEBSCH, *Binäre Formen*, § 51.

The application of the formulæ in CLEBSCH's *Binäre Formen* § 41, gives the concomitants of E_1 as follows:

$$
\begin{aligned}
9i' &= 4I^2 i - 24Ij + 6i^2\,,\\
27j' &= 8I^3 j - 12I^2 i^2 + 36Iij - 72j^2 + 6i^3\,,\\
H_{E_1} &= \tfrac{1}{3}(4j - 2iI + \tfrac{4}{9}I^3)D_1 - \tfrac{1}{6}(\tfrac{4}{3}I^2 - 2i)E_1\,,\\
S_{E_1} &= -J^2 S_{D_1}\,,
\end{aligned}
\tag{5}
$$

where S represents the sextic covariant of the corresponding biquadratic.

The λ-tangents $D_1(\lambda) = 0$ and the μ-tangents $D_2(\mu) = 0$ intersect in the sixteen foci of the cyclics; and from Cayley's theorem it is easy to show that these foci lie by fours on the four director circles. The director circles of f are given by $Y = 0$ (§ 30), where Y is the Jacobian of U', F, G, H; the totality of cyclics, therefore, which have the same director circles as f, is the net

$$t_1 f + t_2 g + t_3 h = 0\,. \tag{6}$$

Included in this net is the system of cyclics confocal with f,

$$f - \rho g - \rho^2 h\,; \tag{7}$$

through each point of the plane there pass in general two orthogonal cyclics of the confocal system, the exceptional points (for which the two fall together) lying on the eight minimal tangents:*

$$g^2 + 4fh = 0 \quad \text{or} \quad D_1(\lambda)D_2(\mu) = 0\,.$$

§ 32. *Contravariants.*—The definition of polar and anti-polar circles given in § 30, may be extended so as to apply to a circle instead of to a point: the polar circle C_1 of a circle C, with respect to a cyclic f, is the locus of points whose polar circles are orthogonal to C; the anti-polar circle C_2 of C is the circle orthogonal to the polar circles of all the points of C. For the normal form the coordinates of these circles are

$$C\colon\ u_i,\quad C_1\colon\ \beta_i u_i,\quad C_2\colon\ \beta_k\beta_l\beta_m u_i\,, \tag{1}$$

from which it can be shown that the polar circle of the anti-polar circle (as also the anti-polar circle of the polar circle) coincides with the original circle. The invariants of (1) are

$$
\begin{aligned}
&I(CC) = \sum u^2 = \sigma',\quad I(CC_1) = \sum \beta_i u_i^2 = -\phi,\quad I(CC_2) = \sum \beta_k\beta_l\beta_m u_i^2 = \chi\,,\\
&I(C_1C_1) = \sum \beta_i^2 u_i^2 = \psi - I\sigma',\quad I(C_1C_2) = K\sigma',\quad I(C_2C_2) = J\chi - K\psi\,.
\end{aligned}
\tag{2}
$$

*See (6) § 32.

Therefore

$\sigma' = 0$ is the quadratic complex of degenerate circles;

$\phi = 0$ is the quadratic complex of circles orthogonal to their polar circles;

$\psi - I\sigma' = 0$ is the quadratic complex of circles whose polar circles are degenerate;

$\chi = 0$ is the quadratic complex of circles orthogonal to their anti-polar circles.

Finally Z, being the Jacobian of σ', ϕ, ψ, χ, represents the circles belonging to the linear system determined by their polars with respect to f, g, h; it consists of the circles orthogonal to the directors circles.

The polar circles of all the points of the plane form the complex $J\chi - K\psi = 0$, while the anti-polar circles form $\psi - I\sigma' = 0$.

Applying the first principle of § 16, the intersections of $C = r_\lambda \rho_\mu$ and $f = a_\lambda^2 a_\mu^2$ are given by

$$L = (fC^2)_{02} = (a\rho)(a\rho')a_\lambda^2 r_\lambda r'_\lambda,$$

$$M = (fC^2)_{20} = (ar)(ar')a_\mu^2 \rho_\mu \rho'_\mu.$$

Therefore, *the complex of circles, which cut f in four points whose anharmonic ratio on the circle is a*, is given by *

$$\frac{i_L^3}{j_L^2} = \frac{i_M^3}{j_M^2} = 24\,\frac{(1 - a + a^2)^3}{(1 + a)^2(2 - a)^2(1 - 2a)^2}. \tag{3}$$

In general the complex is therefore of the twelfth class; but for the equianharmonic and harmonic cases, it reduces to

$$i_L = 0, \quad j_L = 0,$$

which are of the fourth and of the sixth classes respectively; and for the coincident case the complex decomposes into [(4) §16]

$$D_L = i_L^3 - 6j_L^2 = \Phi\sigma'^2.$$

The *tangential complex* Φ, expressed in terms of the fundamental contravariants, is †

$$\Phi = 4(3\sigma'\psi - \phi^2)(3\phi\chi - \psi^2) - (9\sigma'\chi - \phi\psi)^2. \tag{4}$$

The osculating circles belong to both the complexes $i_L = 0$, $j_L = 0$.

* In terms of the fundamental contravariants, this comeplx is $(4\Phi^2 - 6I\sigma'^2)^3 + A\sigma'^3\Phi$, where A is a function of the anharmonic ratio a, and Φ is defined by (4).

† SALMON-FIEDLER, *Raumgeometrie*, 3d edition, vol. II, p. 345.

Each contravariant (§ 16) yields a corresponding covariant—the locus of the vertices of the degenerate circles contained in the complex; thus, from ϕ, ψ, χ, Z we obtain F, G, H, Y. Substituting these in (3), the corresponding covariant is found to be

$$3F^2(G^2 + 4FH);$$

but it is evident that a degenerate circle, whose centre is not on F, can be tangent to F only when one of its minimal lines is tangent to F; so that

$$G^2 + 4FH = 0 \tag{4}$$

represents the minimal tangents of F. These tangents are also given by

$$D_1(\lambda)D_2(\mu) = 0; \tag{5}$$

in fact from the normal forms it can be shown that

$$g^2 + 4fh = -4D_1D_2. \tag{6}$$

§ 33. *Interpretations for the invariants.**—The application of formula (4) § 24 gives the following values of the (2, 2) transvectants:

$$\begin{aligned} &(ff)_{22} = -2I, \qquad (fg)_{22} = 3J, \quad (fh)_{22} = 4K, \\ &(gg)_{22} = I^2 - 4K, \quad (gh)_{22} = \frac{IJ}{2}, \quad (hh)_{22} = \tfrac{3}{4}J^2 - 2IK. \end{aligned} \tag{1}$$

From the conclusion of § 24 we have then:

If $I = 0$, the complex ϕ contains an infinite number of quadruples mutually conjugate with respect to f, and conversely if ϕ contains one such set (and therefore an infinity), $I = 0$; similarly $J = 0$, $K = 0$ are the necessary and sufficient conditions for ϕ containing quadruples mutually conjugate with respect to g and h respectively.

Other interpretations are obtained from the consideration of the covariant quadrics in space. If $I = 0$, the complex ψ contains quadruples of mutually orthogonal circles. If $J = 0$, there exist self-polar point quadruples, i. e., sets of four points possessing the property that the polar circle of each passes through the other three; and χ contains mutually orthogonal quadruples of circles. If $K = 0$, the polar circles of all the points of the cyclic are orthogonal to a fixed circle, which circle is one of the director circles; on this director circle there exist triples of points whose polar circles are mutually conjugate; and χ degenerates into the linear complex (counted twice) of circles orthogonal to the special director circle.

* See also § 37.

§ 34. *Complete binary system.*—In § 28 a system of forms $\sum_3$ of the quadri-quadric

$$f = a_\lambda^2 a_\mu^2 ,$$

was obtained, in terms of which any invariant, concomitant or equi-covariant can be rationally and integrally expressed; if we wish to consider also covariants whose partial orders are unequal, we must add to this system the forms of § 28 and the (1, 0) and (0, 1) transvectants,

$$\begin{aligned} &\rho_1 = (gh)_{01}, \quad \rho_2 = (hf)_{01}, \quad \rho_3 = (fg)_{01}, \\ &\pi_1 = (gh)_{10}, \quad \pi_2 = (hf)_{10}, \quad \pi_3 = (fg)_{10}, \end{aligned} \tag{1}$$

where (§ 13) $\pi_1 = 0$ ($\rho_1 = 0$) represents the loci of points whose polar λ-lines (μ-lines) with respect to g and h coincide; and $\rho_1 = 0$, $\pi_1 = 0$ intersect in the 18 points which have the same conjugates with respect to either f or g, etc. The completeness of the system thus obtained is proved by a combination of the results already obtained, and a comparison with PEANO's system of in- and co-variants; for this comparison it is convenient to express the forms as transvectants.

From (1) §33 the invariants may be written

$$I = -\tfrac{1}{2}(ff)_{22}, \quad J = \tfrac{1}{3}(fg)_{22}, \quad K = \tfrac{1}{4}(fh)_{22} .$$

The covariants are*

$$\begin{aligned} &f, \quad g = (ff)_{11}, \quad h = -(fg)_{11} + \tfrac{1}{4}(ff)_{22} \cdot f, && \text{from (6) § 30;} \\ &\pi_1 = (gh)_{10}, \quad \pi_2 = (hf)_{10}, \quad \pi_3 = (fg)_{10}, && \\ &\rho_1 = (gh)_{01}, \quad \rho_2 = (hf)_{01}, \quad \rho_3 = (fg)_{01}, && \text{from (1);} \\ &D_1 = \tfrac{1}{2}(ff)_{02}, \quad E_1 = \tfrac{1}{2}(gg)_{02}, \quad S_1 = (D_1 E_1)_{10}, && \\ &D_2 = \tfrac{1}{2}(ff)_{20}, \quad E_2 = \tfrac{1}{2}(gg)_{20}, \quad S_2 = (D_2 E_2)_{01}, && \text{from § 31.} \end{aligned}$$

The contravariants (2) § 32 are

$$\begin{aligned} &\sigma' = (CC)_{11}, \quad \phi = -(CC_1)_{11}, \quad \psi = (C_1 C_1)_{11} + I(CC)_{11}, \\ &\chi = (CC_2)_{11}, \quad Z = \text{Jacobian } (\sigma' \phi \psi \chi) ; \end{aligned}$$

where

$$C_1 = (fC)_{11}, \quad C_2 = (hC)_{11} + \frac{J}{4} C, \quad C = r_\lambda \rho_\mu .$$

The invariants and contravariants are complete by § 32, and the completeness of the covariants is shown by their equivalence to PEANO's covariants.† The

*Y is omitted since it is expressible integrally through the other covariants.

† PEANO, Giornale di Matematiche, vol. 20, p. 97, 1882; contravariants are not considered. GORDAN, Mathematische Annalen, vol. 23, p. 388, 1889, gives a complete system of 38 invariants and covariants; but this is reducible to PEANO's 18 forms and therefore also to the forms in the text.

complete binary system consists then of the twenty-three forms:

$$(\Sigma_4)\qquad \begin{array}{llllll} I, & J, & K; & f, & g, & h; \\ \pi_1, & \pi_2, & \pi_3, & \rho_1, & \rho_2, & \rho_3; \\ D_1, & E_1, & S_1, & D_2, & E_2, & S_2; \\ \sigma', & \phi, & \psi, & \chi, & Z. & \end{array}$$

§35. *Equivalence and classification of the cyclics.*—The character of the cyclic:

$$F = A_x^2 = 0, \quad U' = p_x^2 = 0, \tag{1}$$

or

$$f = a_\lambda^2 a_\mu^2 = 0, \tag{2}$$

depends essentially upon the determinant:

$$D(\rho) = |A_{ik} + \rho p_{ik}|, \tag{3}$$

or, by (5) § 29,

$$D(\rho) = \begin{vmatrix} a_{00} & a_{01} & a_{10} & a_{11}+\rho \\ a_{01} & a_{02} & a_{11}-\rho & a_{12} \\ a_{10} & a_{11}-\rho & a_{20} & a_{21} \\ a_{11}+\rho & a_{12} & a_{21} & a_{22} \end{vmatrix}. \tag{4}$$

Applying WEIERSTRASS's theory of quadratic forms * we have:

Two cyclics are equivalent † when and only when their characteristic determinants have proportional elementary divisors.

This condition may be decomposed into a qualitative and quantitative: 1° similarity of the characteristic determinants, i. e., the elementary divisors of both characteristic determinants occur to the same degrees; 2° equality of absolute invariants, i. e., the roots of the characteristic biquadratic

$$D(\rho) = \rho^4 + I\rho^2 + J\rho + K = (\rho-\beta_1)(\rho-\beta_2)(\rho-\beta_3)(\rho-\beta_4) \tag{5}$$

of both characteristic determinants are proportional.

If we call two cyclics of the same "species" when they satisfy the condition 1°, we have in all thirteen species; these, as defined by the symbol ‡ of the char-

* WEIERSTRASS, *Berliner Monatsberichte*, 1868.

† Since every cyclic can be transformed into itself by both proper and improper circular transformations (in general four of each kind), equivalence in the group G and equivalence in the group G' are (for cyclics) identical. Therefore the theorem above applies to both groups.

‡ The notation is that introduced by WEITER in his classification of the quadratic line-complexes, Mathematische Annalen, vol. 7; Ib or [(11)11], for example, signifies that the characteristic determinant has two simple and one double factor, the latter also appearing as a simple factor in all the first minors. The case [(1111)] is omitted as trivial since it can occur only when $f \equiv 0$.

acteristic determinant, are given in the following table:

(6)

	a	b	c	d
I	[1111]	[(11)11]	[(11)(11)]	[(111)1]
II	[211]	[(21)1]	[2(11)]	[(211)]
III	[31]	[(31)]		
IV	[22]	[(22)]		
V	[4]			

In order to study the geometric peculiarities of the different species, it is sufficient to consider them in their normal forms. The quaternary normal forms for a pair of quadrics can be obtained by Weierstrass' methods;* reducing the results to apolar form we obtain the quaternary normal forms of the cyclic as follows:

I
$$\begin{cases} U' = x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ F = \beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \beta_4 x_4^2 & (\beta_1+\beta_2+\beta_3+\beta_4=0); \end{cases}$$

II
$$\begin{cases} U' = x_1^2 + x_4^2 - 2x_2x_3, \\ F = \beta_1 x_1^2 + \beta_2 x_4^2 - 2\beta_3 x_2x_3 + x_2^2 & (2\beta_3+\beta_1+\beta_2=0); \end{cases}$$

III
$$\begin{cases} U' = x_1^2 + x_4^2 - 2x_2x_3, \\ F = \beta_1 x_1^2 + \beta_2(x_4^2 - 2x_2x_3) + 2(\sqrt{2}x_2x_3) & (3\beta_2+\beta_1=0); \end{cases}$$

IV
$$\begin{cases} U' = 2x_1x_4 - 2x_2x_3, \\ F = 2\beta_1 x_1x_4 - 2\beta_2 x_2x_3 + x_3^2 - x_4^2 & (2\beta_1+2\beta_2=0); \end{cases}$$

V
$$\begin{cases} U' = 2x_1x_4 - 2x_2x_3, \\ F = 2\beta_1(x_1x_4 - x_2x_3) + x_2^2 + 2x_1x_3 & (4\beta_1=0). \end{cases}$$

These are the normal forms for Ia, IIa, IIIa, IVa, Va, the remaining species being obtained from these by equating the proper β's.

Applying now the method of § 29, we obtain the following normal forms of the thirteen species of quadri-quadrics:

Ia, $\quad k(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) + l(\lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2) + 4m\lambda_1\lambda_2\mu_1\mu_2$;

Ib, $\quad l(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) - 4m\lambda_1\lambda_2\mu_1\mu_2$;

Ic, $\quad 4\lambda_1\lambda_2\mu_1\mu_2$;

Id, $\quad \lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2 - 2\lambda_1\lambda_2\mu_1\mu_2$;

* Without the use of Weierstrass' general formulæ, corresponding normal forms are obtained in CLEBSCH-LINDEMANN, *Geometrie*, vol. II, p. 202 ff.; for conics compare GUNDELFINGER, *Kegelschnitte*, §16.

IIa, $k(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) + \lambda_1^2\mu_2^2 + 4m\lambda_1\lambda_2\mu_1\mu_2$;

IIb, $\lambda_1^2\mu_2^2 - 2m(\lambda_1\mu_1 - \lambda_2\mu_2)^2$;

IIc, $\lambda_1^2\mu_2^2 + 4m\lambda_1\lambda_2\mu_1\mu_2$;

IId, $\lambda_1^2\mu_2^2$;

IIIa, $k(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2 + 2\lambda_1\lambda_2\mu_1\mu_2) + 2\lambda_1\mu_1(\lambda_1\mu_1 - \lambda_2\mu_2)$;

IIIb, $2\lambda_1\mu_2(\lambda_1\mu_1 - \lambda_2\mu_2)$;

IVa, $\lambda_2^2\mu_1^2 + 4m\lambda_1\lambda_2\mu_1\mu_2 - \lambda_2^2\mu_2^2$;

IVb, $\lambda_2^2(\mu_1^2 - \mu_2^2)$;

Va, $\lambda_1^2\mu_2^2 + 2\lambda_1\lambda_2\mu_1^2$.

From these normal forms a large number of results may be derived, some of which are given in the following table:

		$D(\rho)$	$D_1(\lambda)$	$D_2(\mu)$		
Ia	[1111]	{1111}	{1111}	{1111}	0	general C_{22}
Ib	[(11)11]	{211}	{22}	{22}	2	$C_{11}C_{11}$
Ic	[(11)(11)]	{22}	{22}	{22}	4	$C_{10}C_{10}C_{01}C_{01}$
Id	[(111)1]	{31}	iden.*	iden.*	5	C_{11}^2
IIa	[211]	{211}	{211}	{211}	1	nodal C_{22}
IIb	[(21)1]	{31}	{4}	{4}	3	C_{11} tangent C_{11}
IIc	[2(11)]	{22}	{22}	{22}	3	$C_{11}C_{10}C_{01}$
IId	[(211)]	{4}	iden.*	iden.*	6	$C_{10}^2C_{01}^2$
IIIa	[31]	{31}	{31}	{31}	2	cuspidal C_{22}
IIIb	[(31)]	{4}	{4}	{4}	4	$C_{10}C_{01}$ intersecting on C_{11}
IVa	[22]	{22}	{211}	{22}	2	$C_{12}C_{10}$
IVb	[(22)]	{4}	{4}	iden.*	5	$C_{10}^2C_{01}C_{01}$
Va	[4]	{4}	{31}	{4}	3	C_{12} tangent C_{10}

The first column gives the species; the second, the symbol or "character" of the characteristic determinant; the next three, the characters of the biquadratics D, D_1, D_2; the sixth, the degree of speciality, or the number of conditions necessary in order that a cyclic may belong to the corresponding species; and the last, a description of the form of the curve. For example, consider the species Ib or [(11)11]: the curve consists of two non-degenerate non-tangent circles; the characteristic biquadratic $D(\rho)$ has a double root; each of the discriminants has two double roots; and the species is doubly special. Some conclusions which can be drawn from this table will now be given.

* This means that the corresponding biquadratic vanishes identically.

The species which represent a pair of circles (proper or degenerate) are Ib, Ic, Id, IIb, IIc, IId, IIIb, IVb; for these (and for no others), both D_1 and D_2 are perfect squares. Therefore

The necessary and sufficient condition that $f = 0$ shall represent a pair of circles, is that both discriminants shall be perfect squares.

In terms of the fundamental concomitants the conditions may be expressed thus:

$$3iE_1 + (6j - 2iI)D_1 \equiv 0\,,$$

$$3iE_2 + (6j - 2iI)D_2 \equiv 0\,.$$

In § 31, it was shown that the three biquadratics $D(\rho)$, $D_1(\lambda)$, $D_2(\mu)$, have equal invariants i, j; it does not follow from this that they are equivalent, but it does follow that if their "character" is the same (i. e., if the same equalities hold among their roots), they will be necessarily equivalent. The table then gives the following relations between D, D_1, D_2:*

If the cyclic is irreducible, all three biquadrics are equivalent: in all cases two of the three are equivalent. The necessary and sufficient condition for the equivalence of the discriminants D_1, D_2 is the presence of a linear elementary divisor in the characteristic determinant of f. Translated into geometrical terms, this reads: *except when the cyclic degenerates into either a minimal line and a C_{12} or C_{12}, or into a double minimal line of one system and a pair of minimal lines of the other system—the two pencils of minimal tangents are homographic.*

For the different species, the following relations exist among the invariants:

Ia,	$\delta \neq 0$;
IIa, Ib,	$\delta = 0$;
IIIa, IIb, Id,	$i = 0$, $j = 0$;
IVa, IIc, Ic,	$J \neq 0$, $4K - I^2 = 0$;
Va, IIIb, IVb, IId,	$I = 0$, $J = 0$, $K = 0$.

Therefore $\delta = i^3 - 6j^2 = 0$ is the condition for a node (this may also be obtained from (2) § 16, since for the cyclic there are no P, Q invariants); and $i = 0$, $j = 0$ are the conditions for a cusp.†

* Cf. FROBENIUS, Crelle, vol. 106, pp. 125–188, 1890. The problem of finding the forms when the discriminants are assigned is there studied; geometrically, this is equivalent to finding the system of cyclics with assigned foci.

† The explicit values of i, j have been calculated by CAYLEY, *Two Invariants of the Quadroquadric*, 1893, *Works*, vol. XIII. Cf. *Porism of the in- and circumscribed polygon*, 1870, vol. VIII.

CHAPTER VI.

Double Ternary and other Methods.

§ 36. *Double ternary method.*—In § 10 a minimal line of the plane, or a generator of the fundamental quadric, was defined by a parameter λ, or a binary set $\lambda_1 : \lambda_2$; these coördinates will now be replaced by ternary homogeneous coordinates connected by a quadratic identity. One method of introducing these is to establish a (1, 1) correspondence between the pencil of lines considered and the points of a conic: the trilinear coördinates $r_1 r_2 r_3$ of the points may then be taken as the coördinates of the corresponding lines. Again, by the use of line coördinates, the r's may be defined as follows: let the equations, in line coördinates, of three fixed generators of the first system (or more generally of three fixed linear line-complexes, each of which includes all the generators of the second system) be $l' = 0$, $l' = 0$, $l'' = 0$; the equation of any other generator of the first system is then of the form

$$r_1 l' + r_2 l'' + r_3 l''' = 0 ; \tag{1}$$

these parameters r_i may then be called the ternary coördinates of the line. The quadratic identity (obtained by expressing the fact that (1) is a special linear complex) is

$$R \equiv (l'l')r_1^2 + (l''l'')r_2^2 + (l'''l''')r_3^2 + 2(l'l'')r_1 r_2 + 2(l''l''')r_2 r_3 + 2(l'''l')r_3 r_1 = 0 ,$$

where $(l'l'')$, for example, is the simultaneous invariant of l' and l''.

If we introduce ternary coördinates for each pencil of minimal lines or generators in either of these ways, a point may be defined by the six coördinates of the two minimal lines passing through it,

$$r_1 : r_2 : r_3 , \quad s_1 : s_2 : s_3 , \tag{2}$$

connected by two quadratic identities

$$R = 0 , \quad S = 0 . \tag{3}$$

The group G_6 then consists of those linear transformations of the r's and s's, which preserve the identities (3). See § 10.

A curve whose partial orders are $2m$, $2n$ (so that the corresponding binary form is $a_\lambda^{2m} a_\mu^{2n}$) is represented in the present coördinates by an equation of the form:

$$f'(\overbrace{r_1, r_2, r_3}^{m} ; \quad \overbrace{s_1, s_2, s_3}^{n}) = 0 . \tag{4}$$

From (3) the same curve is represented by

$$f' + MR + NS = 0, \tag{5}$$

where M, N are any double ternary forms of orders $(m-2, n)$ and $(m, n-2)$, respectively; this lack of correspondence between the curves and the forms may be removed, as in § 3, by the introduction of apolarity:

Of the system of forms $f' + MR + NS$, one and only one, (f), is identically apolar to R and S.

Combining this result with that obtained above for G_6 we have:

The inversion theory of a curve $C_{2m, 2n}$ is equivalent to the study of the forms:

$$f = b_r^m \beta_s^n, \quad R = p_r^2, \quad S = q_s^2, \tag{6}$$

where f is identically apolar to R and S, and the ternary variables r, s undergo independent linear transformations.

If r and s are considered as homogeneous point coördinates in distinct planes, $R = 0$ represents a conic in the first plane, $S = 0$ a conic in the second plane, and (4) a reciprocity of orders (m, n) between the two planes; i. e., to each point r of the first plane corresponds, by means of $f' = 0$, a curve K_r of order n in the second plane, and to each point s of the second plane, a curve K_s of order m in the first plane. Between the points of the two conics, the reciprocity $f' = 0$, establishes a $(2m, 2n)$ correspondence as follows: to each point of the conic R corresponds, on the conic S, the $2n$ points cut out by the curve K_r, etc. The same correspondence being established by all the forms (4), the theorems above may be restated thus:

Of all the reciprocities (m, n) which establish the same $(2m, 2n)$ correspondence between the points of two conics R, S, one and only one possesses the property that all the curves K_r are apolar to S, and all the curves K_s apolar to R.

The inversion geometry of a $C_{2m, 2n}$ has for an equivalent the study of the reciprocity (m, n) between two planes, in connection with a conic in each plane—the reciprocity being apolar to each of the conics, and the planes undergoing independent projective transformations.

§ 37. *Application to the cyclic.*—The above method is limited to the case where the partial orders of the curve are even, so that the simplest application is to the cyclics. The forms (6) § 36 are in this case

$$f = \sum b_{ik} r_i s_k = b_r \beta_s, \quad R = p_r^2, \quad S = q_s^2, \tag{1}$$

and no reduction to apolar form is necessary. The last theorem of the preceding article takes the form:

The theory of a correlation between two planes, with reference to two fundamental conics in those planes, gives a representation of the inversion geometry of the cyclic.

Let u_i, v_i denote line coördinates in the two planes, and B_{ik} the minor of b_{ik} in $|b_{ik}|$; then the correlation

$$f = \sum b_{ik} r_i s_k = 0$$

may also be written in any of the forms:

$$(2) \qquad u_i = \sum_k b_{ik} s_k, \quad v_i = \sum_k b_{ki} r_k, \quad s_i = \sum_k B_{ki} u_k, \quad r_i = \sum_k B_{ik} v_k,$$

and the two points of S, which correspond to a point r on R, are cut out by the line v given by the second of these representations.

The correlation transforms R into another conic S', and S into R', so that in each plane we have a pair of conics by whose aid the fundamental points of the (2, 2) correspondence are determined as follows:

The "branch points" of R (i. e., *the points for which the corresponding s points coincide*) *are $L_1L_2L_3L_4$, the intersections of R and R'; and similarly the branch points $M_1M_2M_3M_4$ of S are the intersections of S and S'. The "double points" of R* (i. e., *the points corresponding to the branch points of S*), *$L'_1L'_2L'_3L'_4$, are the points of contact on R of the tangents common to R and R'; the double points of S, $M'_1M'_2M'_3M'_4$, are obtained similarly from the tangents common to S and S'.*

The four sets of points thus obtained represent the biquadratics D_1, D_2, E_1, E_2 of § 31; and the above construction furnishes a simple synthetic proof of CAYLEY's theorem (§ 31): the pair of conics S', S being correlative to both R, R' and S, S', it follows that the pairs R, R' and S, S' are homographic, and therefore the homologous sets of points $L_1L_2L_3L_4$, $M_1M_2M_3M_4$ (representing the discriminants D_1D_2) have equal anharmonic ratios on the conics R and S.

The ternary forms of the concomitants of the cyclic are most conveniently obtained from a normal form of the system (1). The binary normal form [C] § 28, is

$$(3) \qquad f = k(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) + l(\lambda_1^2\mu_2^2 + \lambda_2^2\eta_1^2) + 4m\lambda_1\lambda_2\mu_1\mu_2.$$

Putting

$$r'_1 = \lambda_1^2, \; r'_2 = 2\lambda_1\lambda_2, \; r'_3 = \lambda_2^2; \quad s'_1 = \mu_1^2, \; s'_2\, 2\mu_1\mu_2, \; s'_3 = \mu_2^2;$$

so that the identities are

$$R = r'^2_2 - 4r'_1r'_3 = 0, \quad S = s'^2_2 - 4s'_1s'_3,$$

equation (3) becomes

$$f = k(r'_1s'_1 + r'_3s'_3) + l(r'_1s'_3 + r'_3s'_1) + mr'_2s'_2;$$

and substituting

$$2ir'_1 = r_1 + ir_3, \quad r'_2 = r_2, \quad 2ir'_3 = r_1 - ir_3, \quad \text{etc.},$$

we have the required ternary normal form:

$$[D] \qquad \begin{aligned} f &= b_1 r_1 s_1 + b_2 r_2 s_2 + b_3 r_3 s_3 , \\ R &= r_1^2 + r_2^2 + r_3^2 , \quad S = s_1^2 + s_2^2 + s_3^2 , \end{aligned}$$

where

$$(3) \qquad 2b_1 = -k - l , \quad b_2 = m , \quad 2b_3 = k - l .$$

From the concomitants of $[C]$ given in §§ 28, 31, and the transformation formulæ above, the concomitants of $[D]$ are found to be

$$I = -2\sum b_i^2 , \quad J = -8 b_1 b_2 b_3 , \quad K = \sum b_i^4 - 2\sum b_i^2 b_k^2$$

(or instead of K we may consider $K' = I^2 - 4K = 16\sum b_i^2 b_k^2$);

$$f = \sum b_i r_i s_i , \quad g = -2\sum b_k b_l r_i s_i , \quad h = \sum b_i (b_i^2 - b_k^2 - b_l^2) r_i s_i ;$$

$$D_1 = \sum b_i^2 r_i^2 , \quad E_1 = 4\sum b_k^2 b_l^2 r_i^2 , \quad S_1 = (b_1^2 - b_2^2)(b_2^2 - b_3^2)(b_3^2 - b_1^2) r_1 r_2 r_3 ;$$

$$D_2 = \sum b_i^2 s_i^2 , \quad \text{etc.}$$

The conics R, R', S, S' are covariants of $[D]$; their concomitants are therefore also concomitants of $[D]$, and expressible in terms of those just written. It is sufficient to consider one pair:

$$(4) \qquad R = \sum r_i^2 , \quad R' = \sum b_i^2 r_i^2 .$$

The invariants of these conics are

$$\Delta = 1 , \quad \theta = \sum b_i^2 , \quad \theta' = \sum b_i^2 b_k^2 , \quad \Delta' = b_1^2 b_2^2 b_3^2 ;$$

and the covariants are

$$R , \quad R' , \quad R'' = \sum b_k^2 b_l^2 r_i^2 , \quad R''' = (b_1^2 - b_2^2)(b_2^2 - b_3^2)(b_3^2 - b_1^2) r_1 r_2 r_3 .$$

The expressions in terms of the fundamental forms are then,

$$(5) \quad \theta = -\tfrac{1}{2} I , \quad \theta' = \tfrac{1}{16} K' , \quad \Delta' = {}_6{}^1{}_4 J^2 , \quad R' = D_1 , \quad R'' = \tfrac{1}{4} E_1 , \quad R''' = S .$$

From these relations we may obtain interpretations for the invariants of the cyclic of a different character from those given in §33.

If $I = 0$, then $\theta = 0$ from (5), i. e., R' is harmonically circumscribed as to R; combining this with the construction of the double points given above, and LINDEMANN'S interpretation of the Hessian, we see that †

$I = 0$ is the condition that the double points shall form the Hessian quadruple of the branch points.

* SALMON-FIEDLER, *Kegelschnitte*, Chapter 19.

† LINDEMANN, Bulletin Société Mathématique de France, vol. 5, p. 119, 1876.

Similarly, since $\theta' = 0$ if $K' = 0$, it follows that

$K' = 0$ is the condition that the branch points shall form the Hessian quadruple of the double points.

Finally $\Delta = 0$ if $J = 0$, so that the conic R' degenerates into a pair of lines, and the conics R, R' have only two common tangents, and therefore

If $J' = 0$, the four double points reduce to two; and the pairs of points on R which correspond to the points of S, form an involution of which the two double points are the foci.

§ 38. *A series of extensions.*—An important advance in the theory of binary forms is marked by the appearance in 1866 of HESSE's "Uebertragunsprincip."* This principle, together with the notion of apolarity, has been the source of a series of methods whose best presentation in invariant form has probably been given by FRANZ MEYER in his book, *Apolarität und rationale Curven* (1883).† The basis of these methods may be stated: the theory of binary forms, or point groups on a line, is (roughly) equivalent to the projective theory of surfaces (i. e., $k-1$ dimensional manifolds) in a space P_k of k dimensions, with reference to a fundamental rational curve N_k of the kth order to which the surfaces are all apolar. (The possession of an apolar N_k, of course in general specializes the surfaces considered.)

The partial extension of this principle to double binary forms or the inversion geometry of curves, which was given in § 36, may be generalized thus:

The geometry of a $C_{m,n}$ where $m = hm'$, $n = kn'$, has for an equivalent the theory of an (m', n') reciprocity,

$$(a_1 r_1 + a_2 r_2 + \cdots + a_{h+1} r_{h+1})^{m'} (a_1 s_1 + a_2 s_2 + \cdots + a_{k+1} s_{k+1})^{n'} = 0,$$

between two spaces, P_h, P_k, with reference to fundamental curves N_h, N_k in those spaces—the reciprocity being identically apolar to each of the curves.

This principle gives as many methods of treating a $C_{m,n}$ as there are divisors h, k of m, n. Two choices which are always possible are 1°, $h = 1$, $k = 1$; 2°, $h = m$, $k = n$. In 1° the spaces P_h, P_k are one-dimensional, and the curves N_h, N_k need not be considered, since they coincide with the spaces; the above principle reduces them to a statement of the double binary method, the equivalent of the geometry of a $C_{m,n}$ being the theory of the double binary form $a_\lambda^m a_\mu^n$. In 2°, $m' = 1$, $n' = 1$, the reciprocity is bilinear, and the condition of apolarity is superfluous: the principle then gives as a representation of the geom-

* Crelle, 1866.

† Also in four memoirs, Mathematiche Annalen, vol. 21, 1883, especially, *Ein neues Uebertragunsprincip für binäre Formen deren Ordnungszahl ein nicht prime ist.* Cf. LINDEMANN, *Ueber die Darstellung binärer Formen und ihrer Covarienten durch geometrishe Gebilde im Raume,* Mathematische Annalen, vol. 23, pp. 111-142, 1884.

etry of a $C_{m,n}$, the projective theory of a linear reciprocity between spaces of dimensionalities h, k, with reference to fixed rational curves of orders h, k in those spaces.

The characteristic feature of the methods obtained in this chapter (including as a special case the double binary) is that they are concerned with two distinct spaces or with double forms, thus differing essentially from the quaternary method. The connections between the double binary and quaternary methods, which were considered in chapter IV, admit of generalization to double ternary and higher methods; and the same is true of most of the results in chapter I.

COLUMBIA UNIVERSITY.

A SIMPLE PROOF OF THE FUNDAMENTAL CAUCHY-GOURSAT THEOREM*

BY

ELIAKIM HASTINGS MOORE

Introduction.

GOURSAT in two memoirs (Acta Mathematica, vol. 4, 1884; Transactions of the American Mathematical Society, vol. 1, 1900) has proved CAUCHY'S integral theorem:

$$\int_C f(z)dz = 0,$$

without the assumption of the continuity of the derivative $f'(z)$ on the closed region R bounded by the curve of integration C, and thereby he has laid deeper foundations for the CAUCHY-RIEMANN theory of functions of the complex variable. An abstract of these memoirs is to be found in the Bulletin of this Society for June, 1899, pp. 427–429.

GOURSAT set out by a direct process† to evaluate the integral in question. In the present paper, by an indirect process, I prove that the integral has the value 0. The essential elements of the proof are those of GOURSAT'S first paper; by the modification indicated, and by the imposition on the curve C of a certain condition fulfilled by all the usual curves, one avoids the necessity of introducing the lemma to which GOURSAT'S second paper is devoted.

The necessary preliminary definitions and theorems are given in some detail in § 1, in which connection I refer especially to JORDAN'S *Cours d'Analyse*, 2d ed., vol. 1, 1893, and to HURWITZ'S address at the Zurich Congress of 1897 entitled: *Über die Entwickelung der allgemeinen Theorie der analytischen Functionen in neuerer Zeit* (*Verhandlungen des ··Mathematiker Kongresses in Zürich* ···; Teubner, 1898). Then in §§ 2 and 3 I state and prove the two

* Presented to the Society June 29, 1900, at the New York meeting. Received for publication October 30, 1900.

† STOLZ (*Grundzüge der Differential- und Integralrechnung*, vol. 2, p. 218, 1896) states that this process of GOURSAT'S first memoir was an improvement of an older method; his reference to LAURENT, *Théorie des résidus*, 1895, p. 38, I have not succeeded in utilizing.

principal CAUCHY-GOURSAT theorems corresponding to the two principal forms * of CAUCHY'S theorem.

§ 1. *Preliminary definitions and theorems.*

1°. The following notations are useful in abbreviating general statements involving the notion of limits. With respect to a set S of points in the plane of the complex variable z and a point $z = a$ of the plane, the notation S_a indicates the set S with the exception of the point a if a is of the set S, and the notation $S_{a\eta}$, where η is a positive real number, denotes the subset of S_a lying within the circle (a, η) of center a and radius η. The notations may be read thus: the set S a apart; the set S a apart within the η-neighborhood of a.

2°. The point $z = a$ is a limit point of the set S if every set $S_{a\eta}$ contains one point and so an infinitude of points. The set is closed, if it contains all its limit points.

If a set S is closed and contains an infinite sequence of subsets S_ν, each subset S_ν containing an infinitude of points of the set S and containing also the succeeding subset $S_{\nu+1}$, then the subsets S_ν have at least one point of S in common; and if further the S_ν in any way lie respectively in squares forming a sequence of squares with indefinitely decreasing area, then the S_ν have only one point in common.

3°. A (real or complex) function $f(z)$ is uniquely defined for every point z of a set S with a limit point $z = a$.

The function $f(z)$ has on the z-set S for $z = a$ the limit f, in symbols,

$$\lim_{\substack{z \mid S \\ z=a}} f(z) = f,$$

if for every positive number ϵ there exists a positive number δ_ϵ such that

$$|f(z) - f| < \epsilon \qquad (z|S_{a\,\delta_\epsilon}),$$

for every point z of the set $S_{a\,\delta_\epsilon}$.

The point $z = a$ being of the set S, at $z = a$ on the set S the function $f(z)$ is continuous if

$$\lim_{\substack{z \mid S \\ z=a}} f(z) = f(a),$$

and it has the (finite) derivative $f'(a)$ if

* Reference should also be made to the following expositions of Cauchy's theorem:
PRINGSHEIM: Münchner Sitzungsberichte, vol. 25, pp. 39–72, 295–304, 1895;
BÔCHER: Bulletin of the American Mathematical Society, ser. 2, vol. 2, pp. 146–149, 1896;
STOLZ: loc. cit., p. 217 ff., 1896.
The papers of PRINGSHEIM contain interesting historical data concerning the theorem.

$$\mathop{\mathbf{L}}_{\substack{z \mid S \\ z=a}} \frac{f(z)-f(a)}{z-a} = f'(a),$$

that is, if for the function $\Delta(z)$ defined on the set S by the identity:

$$f(z) = f(a) + (z-a)f'(a) + \Delta(z),$$

there is for every ϵ a δ_ϵ such that

$$|\Delta(z)| < \epsilon\,|z-a| \qquad (z|S,\ |z-a| < \delta_\epsilon)$$

for all points z of the set S within the δ_ϵ-neighborhood of a.

4°. An equation: $z = F(t)$, where $F(t)$ is a complex single valued function of the real variable t on the interval $t_0 \cdots T$, defines in the z-plane a curve C,—a path* of the point z from $z_0 = F(t_0)$ to $Z = F(T)$.

The curve C is continuous when the function $F(t)$ is continuous.

The arc z_0Z of the curve C has as length † the limit, if uniquely existent, of the length of the broken line $z_0z_1z_2 \cdots z_nZ$ whose vertices correspond to the $n+2$ points $t_0t_1t_2 \cdots t_nT$ of a partition of the interval t_0T into $n+1$ subintervals—the limit as the lengths of all subintervals indefinitely decrease with the indefinite increase of n. If the arc z_0Z has a length, so does its every arc-part.

A rectifiable curve is a continuous curve with length.

It is easy to see in how far the path-curve in definition and properties is independent of the particular parameter t.

5°. The definite integral of a function $f(z)$ defined upon a path C along that path has the definition:

$$\int_C f(z)\,dz = \mathop{\mathbf{L}}_{\delta_\kappa=0} \sum_{\kappa=0}^{n} f(\zeta_\kappa)(z_{\kappa+1} - z_\kappa),$$

where z_{n+1} is Z, and ζ_κ is any point on the arc $z_\kappa z_{\kappa+1}$, and δ_κ is $t_{\kappa+1} - t_\kappa$.

This ‡ definite integral exists if the path C is rectifiable and the function $f(z)$ is continuous on C (JORDAN: loc. cit., §§ 193–196, 1893).

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* For present purposes we need to consider only paths lying entirely in the finite plane and containing their extreme points.

† SCHEEFER: Acta Mathematica, vol. 5, pp. 49–82, 1884; ASCOLI (Cf. Jahrbuch der Mathematik, vol. 16, p. 339, [1884]); JORDAN: loc. cit., §§ 105–113, 1893; STUDY: Mathematische Annalen, vol. 47, pp. 298–316, 1896.

‡ The two real or complex functions $P(z)$, $Q(z)$ being defined upon a path C, the definite integral of the binomial differential expression $P(z)dx + Q(z)dy$ has the similar definition:

$$\int_C P(z)dx + Q(z)dy = \mathop{\mathbf{L}}_{\delta_\kappa=0} \sum_{\kappa=0}^{n} P(\zeta_\kappa)\,(x_{\kappa+1} - x_\kappa) + \mathop{\mathbf{L}}_{\delta_\kappa=0} \sum_{\kappa=0}^{n} Q(\zeta_\kappa)\,(y_{\kappa+1} - y_\kappa).$$

By a proof similar to that cited above one has the theorem: this definite integral exists if the path C is rectifiable and the functions $P(z)$, $Q(z)$ are continuous on C. I take occasion to refer to this existence theorem since it seems to be neither in JORDAN'S *Cours d'Analyse* nor in PICARD'S *Traité d'Analyse*, and since PRINGSHEIM (loc. cit., pp. 48–55, 1895) carefully expounds for a certain class of curves a definition and an existence theorem which are in fact special cases of the definition and theorem just given. PRINGSHEIM seems to be unfortunately out of touch with the current notion of the general rectifiable curves (Cf. pp. 48, 49, 55, 59, 60).

6°. Of a set S a point a of the plane is (α) an inner point, (β) an outer point, or (γ) a boundary point, according as (α, β) there exists a neighborhood of a, (α) all of whose points belong to S, (β) none of whose points belong to S, or (γ) every neighborhood of a contains at least one point of S and one point not of S.

A set S is connected if any two of its points z_0 Z may be joined by a broken line $z_0 z_1 \cdots z_n Z$ of a finite number of links whose extremities $z_1 \cdots z_n$ are inner points of S and each of length less than a preassigned length.

A set connected and made up exclusively of inner points is with HURWITZ called a continuum.

7°. A simply * closed continuous curve is a continuous curve whose extreme points coincide but whose points are otherwise distinct. These curves have been carefully studied by JORDAN (loc. cit., §§ 96–113).

Such a curve serves as the boundary between two regions of the plane, the exterior region and the interior region. These regions are continua; after the addition of the curve each region remains connected.

8°. A continuum is simply connected if every simply closed continuous curve lying in the continuum has its interior region lying entirely in the continuum. Every continuous curve lying in a simply connected continuum lies in the interior of a simply closed continuous curve lying in the same continuum. One readily constructs such a curve in connection with a sufficiently fine grating of squares superposed on the continuum.

§ 2. *The Cauchy-Goursat theorem for integrals over the boundary of simply connected regions.*

THEOREM. *The definite integral*

$$\int_C f(z)dz$$

exists and has the value 0, *if*

1) *the path of integration C is a simply closed continuous rectifiable curve met by the various lines parallel to the xy axes in the z-plane $(z = x + iy)$ in a finite number of points and segments of coincidence, and moreover having the property* 2);

2) † *for every point ζ of C, if a square with sides parallel to the axes converges in any way to the point ζ, the ratio of the total length of the arcs of C*

* HURWITZ'S "simply" is preferable to JORDAN'S "without multiple point."

†As indicated in the introduction the theorem with the omission of this condition on the curve of integration remains true. It would then be interesting to inquire whether this property of the curve follows from those of hypothesis 1). The usual curves of integration obviously satisfy the condition 2) for the value $\rho_\zeta = 1$. In particular, the proof here given is sufficient to extend the CAUCHY theorem as given by JORDAN to the corresponding CAUCHY-GOURSAT form (Cf. § 3).

lying on the square to the perimeter of the square is ultimately less than a certain constant ρ_ζ, which may vary as ζ traverses C;

3) *on the region R, consisting of the curve C and its interior region, the integrand function $f(z)$ is a single valued continuous function of z with a single valued derivative $f'(z)$.*

In the first place, the integral

$$J = \int_C f(z)dz$$

exists (§ 1, 5°).

We are to prove then that $J = 0$. The proof * is indirect. We set

$$J = J_0, \quad |J_0| = \eta_0,$$

and suppose that $\eta_0 > 0$.

The region $R = R_0$ lies within some square S_0 with sides parallel to the xy axes and of length γ_0. By the introduction of two diameters we subdivide this square S_0 into four equal squares S_1', S_1'', S_1''', S_1^{iv}. In view of hypothesis 1) we admit the usual dissection arguments. The region R is subdivided into a finite number of regions bounded by simply closed continuous curves pieced together from pieces of C_0 and of the two diameters of S_0. These regions lie each in one of the smaller squares. We denote for brevity the lot of regions in S_1' by R_1', and by C_1' the lot of bounding curves described each in the sense dictated by the sense of description of C_0, and by

$$J_1' = \int_{C_1'} f(z)dz \qquad (|J_1'| = \eta_1')$$

the sum of the integrals of $f(z)dz$ over those path-curves, and so for S_1'', S_1''', S_1^{iv}.

Then we have

$$J_0 = J_1' + J_1'' + J_1''' + J_1^{\text{iv}}, \qquad 0 < \eta_0 \leqq \eta_1' + \eta_1'' + \eta_1''' + \eta_1^{\text{iv}}.$$

Hence one of the four η_1's must be $\geqq \eta_0/4$. We denote this † η_1 by the notation η_1 without superscript, and so for the corresponding J_1, C_1, R_1, S_1. Hence in the square S_1 we have a state of affairs similar to that initially existing in the square S_0, except that R_1 consists of several regions of the type R_0 for which we are considering the sum of the corresponding integrals each in the sense specified,— a fact however which does not interfere with the indefinite repetition of the dissection process.

* The proof is in spirit indirect, although it may immediately be cast in the direct form,— a fact remarked by my colleague Mr. MASCHKE.

† In case of ambiguity the choice is a matter of indifference.

Thus for every integer ν there is a square S_ν of length $\gamma_0/2^\nu$; S_ν contains R_ν consisting of a finite number of regions, bounded each by a simply closed continuous curve; these curves each described in the proper sense form the complete boundary C_ν of R_ν; the integral

$$J_\nu = \int_{C_\nu} f(z)dz$$

is the sum of the integrals of $f(z)dz$ over these curves so described; further,

$$|J_\nu| = \eta_\nu \geqq \frac{\eta_0}{4^\nu}.$$

This dissection process determines a definite point $z = \zeta$ which lies on every square S_ν $(\nu = 0, 1, 2, \cdots)$. It is therefore a limit point of the region R, — since every R_ν contains in fact a (simple or compound) curve of integration C_ν, — and it is, indeed, since R is closed, a point of R, either an inner point or a point of the boundary C. For every point z of the square S_ν one has

$$|z - \zeta| \leqq 2\gamma_0 2^{-\nu}.$$

We set for points z of the region R

$$f(z) = f(\zeta) + (z - \zeta)f'(\zeta) + \Delta(z).$$

Introducing a positive number ϵ subject to later determination we have, by hypothesis 3) in accordance with 3° of §1, a positive number δ_ϵ such that on the region R within the circle (ζ, δ_ϵ)

$$|\Delta(z)| < \epsilon\,|z - \zeta| \qquad (z|R;\ |z-\zeta| < \delta_\epsilon).$$

Choosing ν_ϵ so that the square S_{ν_ϵ} lies within the circle (ζ, δ_ϵ), e. g., so that

$$2\gamma_0 2^{-\nu_\epsilon} < \delta_\epsilon,$$

we have on every region R_ν $(\nu \geqq \nu_\epsilon)$

$$|\Delta(z)| < \epsilon\,|z - \zeta| \leqq 2\epsilon\gamma_0 2^{-\nu} \qquad (z|R_\nu,\ \nu \geqq \nu_\epsilon).$$

Then on integrating the trinomial expression for $f(z)dz$ along the various path-curves making up C_ν $(\nu \geqq \nu_\epsilon)$, remembering that over such closed curves $\int dz = 0$, $\int z dz = 0$, and applying the usual mean value theorem, we have

$$J_\nu = \int_{C_\nu} \Delta(z)dz; \qquad |J_\nu| = \eta_\nu < 2\epsilon\gamma_0 2^{-\nu}\lambda'_\nu \qquad (\nu \geqq \nu_\epsilon),$$

where λ'_ν is the total length of the curves of C_ν. Then

$$\lambda'_\nu \leqq 4\gamma_0 2^{-\nu} + \lambda_\nu,$$

where λ_ν is the total length of the arcs of the original curve C on the square S_ν of perimeter $4\gamma_0 2^{-\nu}$. Hence, by hypothesis 2), for all ν's greater than a certain* ν_0

$$\lambda_\nu < 4\gamma_0 2^{-\nu}\rho \qquad (\nu > \nu_0),$$

where ρ is a certain positive constant. Thus

$$\eta_\nu < 8\epsilon\gamma_0^2 4^{-\nu}(1+\rho) \qquad (\nu \geqq \nu_\epsilon\,;\ \nu > \nu_0),$$

while

$$\eta_\nu \geqq \eta_0 4^{-\nu} \qquad (\nu = 0,\ 1,\ 2,\ \cdots).$$

Thus by the mediation of a properly chosen ν one has the relation

$$8\epsilon\gamma_0^2(1+\rho) > \eta_0,$$

which is independent of ν. But the positive number ϵ remains at our disposal and may now be chosen so that

$$8\epsilon\gamma_0^2(1+\rho) < \eta_0,$$

since $\eta_0 > 0$. Thus we reach the contradiction needed to complete the proof of the theorem.

§ 3. *The Cauchy-Goursat theorem for integrals over a closed rectifiable curve lying on a continuum.*

We may with HURWITZ (loc. cit., pp. 101–103) formulate the CAUCHY theorem as proved by JORDAN (loc. cit., §§ 196–198) as follows:

THEOREM. *The definite integral*

$$\int_C f(z)d(z)$$

exists and has the value 0, *if the path of integration C is a closed continuous rectifiable curve lying on a simply connected continuum on which the integrand function f(z) is synectic.*

A function $f(z)$ existing as a single valued function of z on a continuum is synectic if for every such z it is continuous and has a derivative in each axial direction, these axial derivatives being equal, and this common derivative-value being itself a continuous function on the continuum, or, what is the same thing, if for every such z it is continuous and has a single valued derivative $f'(z)$ itself a continuous function on the continuum.

In JORDAN'S proof the theorem is shown to depend upon the special case for a triangle lying on the simply connected continuum. In this reduction the integrand function $f(z)$ enters only with its property of continuity.

* If ζ is an inner point of R, then the R_ν ultimately lie within R and the λ_ν are ultimately 0.

Accordingly by the use of the GOURSAT proof or that of § 2 for this case of the triangle we may enunciate the theorem referred to in the caption of § 3.

THEOREM. *The definite integral*

$$\int_C f(z)d(z)$$

exists and has the value 0, *if the path of integration* C *is any closed continuous rectifiable curve lying on a simply connected continuum on which the integrand function* $f(z)$ *is a single valued continuous function of* z *with a single valued derivative* $f'(z)$.

THE UNIVERSITY OF CHICAGO.